Standard Finite Elements and Weighted Regularization

A Rehabilitation

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Slides of the talk given by MC in Kiel, 26 Jan. 2001 and by MD in Orsay, 15 March 2001.

Outline

- **Eigen-problem for Maxwell equations. "Regularization" by the interior term**

$$(u, v) \longmapsto \int_\Omega s \; \mathrm{div}\, u \; \mathrm{div}\, v \; \mathrm{d}x.$$

- **Reentrant corners: Non-H^1 singularities cannot be approximated by nodal elements.**

- **Boundary penalization: A lesson on theory *vs* practice**

$$(u, v) \longmapsto \lambda \int_{\partial\Omega} (u \times n) \cdot (v \times n) \; \mathrm{d}\sigma.$$

- **Weighted regularization: Convergence restored**

- **Convergence rates**

Maxwell eigenvalue problem

Permittivity ε and permeability μ. Find non-zero ω such that $\exists\ (E,H) \neq 0$

(Maxwell)
$$\begin{cases} \operatorname{rot} E - i\omega\,\mu H = 0 \quad \& \quad \operatorname{rot} H + i\omega\,\varepsilon E = 0 \quad \text{in} \quad \Omega, \\ E \times n = 0 \quad \& \quad H \cdot n = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

(perfect conductor boundary conditions).

Homogeneous and isotropic medium: ε, μ constant > 0. May assume $\varepsilon = \mu = 1$.

$$\Longrightarrow \qquad \operatorname{div} E = 0 \quad \& \quad \operatorname{div} H = 0$$

An “electric" variational formulation: Find non-zero ω such that

$\exists\ E \in H_0(\operatorname{rot}\,;\,\Omega) \setminus \{0\}$ **solves** $\forall \widetilde{E} \in H_0(\operatorname{rot}\,;\,\Omega)$

$$(1) \qquad \int_\Omega \operatorname{rot} E \cdot \operatorname{rot} \widetilde{E} = \omega^2 \int_\Omega E \cdot \widetilde{E}$$

Non-elliptic bvp: ∞-dim. e-space for $\omega = 0$ (does not satisfy $\operatorname{div} E = 0$).

A first strategy : edge elements

The infinite dimensional eigen-space $\mathfrak{E}_0$ for $\omega = 0$ is

$$\mathfrak{E}_0 = \{E = \operatorname{grad} \varphi \mid \varphi \in H^1_0(\Omega)\}.$$

Unless divergence-free elements are used (?), the space $\mathfrak{E}_0$ has an influence on any discrete scheme. The spurious modes are also approximated. If they are approximated by non-zero eigen-frequencies, they pollute the whole spectrum.

Generalized pollution is avoided by the use of *spurious free elements* (if they exist !), that is, discrete spaces where the 0 e-value is approached by exact 0.

Such elements do exist:

they are realized by the 2 generations of NEDELEC's edge elements (1980 and 1986).

These discrete families are rot conforming but *not div conforming*.

The compatibility conditions between two neighboring elements are obtained via

moments across the common edge.

An alternative strategy : Nodal FEMs need Regularization

The electric field E solution of the Maxwell e-value problem belongs to

$$X_N = \Big\{ u \in L^2(\Omega)^3 \;;\; \operatorname{rot} u \in L^2(\Omega)^3,\; \operatorname{div} u \in L^2(\Omega),\; u \times n = 0 \text{ on } \partial\Omega) \Big\}.$$

Reintroduce the divergence via a regularization parameter $s > 0$, i.e. add

$$s \langle \operatorname{div} E, \operatorname{div} \widetilde{E} \rangle .$$

$\Longrightarrow$ Variational formulation in X_N. For $s > 0$, find non-zero $\omega[s]$

$\exists\, E \in X_N \setminus \{0\}$ solves $\forall \widetilde{E} \in X_N$

$$\int_\Omega \operatorname{rot} E \cdot \operatorname{rot} \widetilde{E} + s \operatorname{div} E \ \operatorname{div} \widetilde{E} = \omega[s]^2 \int_\Omega E \cdot \widetilde{E} \ .$$

The solutions ω independent of s are the Maxwell e-frequencies.

Spurious eigenfrequencies: $\omega[s]^2 = s\nu$ with Dirichlet eigenvalues ν.

Elliptic bvp: $s = 1$: Laplacian; $s > 1$: Lamé with $\mu = 1, \lambda = s - 2$.

Coercive bilinear form; stability and convergence of FEM. . . However . . . /. . .

A topological barrier

Any X_N-conforming finite element space $\mathfrak{X}_N^h$ is contained in $\mathcal{C}^0(\overline{\Omega})$
$\Longrightarrow$ $\mathfrak{X}_N^h$ is H^1-conforming
$\Longrightarrow$ The solution of the Galerkin approximation in $\mathfrak{X}_N^h$, $E_h \in H^1(\Omega)^3$ thus to

$$H_N := X_N \cap H^1(\Omega)^3.$$

Let $H_N^\infty := \mathcal{C}^\infty(\overline{\Omega})^3 \cap X_N$. For $u \in H_N^\infty$ there holds

$$\int_\Omega |\operatorname{rot} u|^2 + |\operatorname{div} u|^2 = \int_\Omega |\operatorname{grad} u|^2.$$

Thus the closure $\overline{H}_N^\infty$ of H_N^∞ for the norm $\left(\|u\|^2 + \|\operatorname{rot} u\|^2 + \|\operatorname{div} u\|^2\right)^{1/2}$ is contained in H_N. In fact

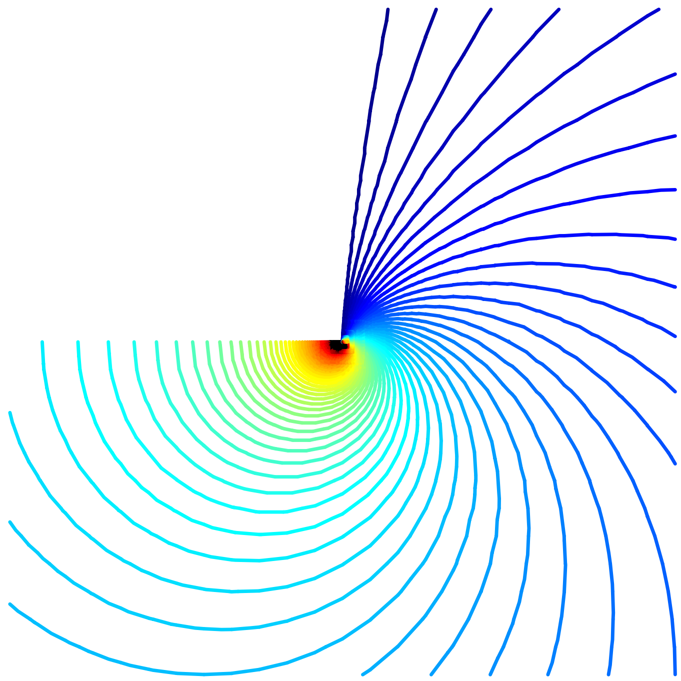
$$\boxed{\overline{H}_N^\infty = H_N}$$

$\Longrightarrow$ E_h converges to $E_{\mathrm{wr}} \in H_N$.

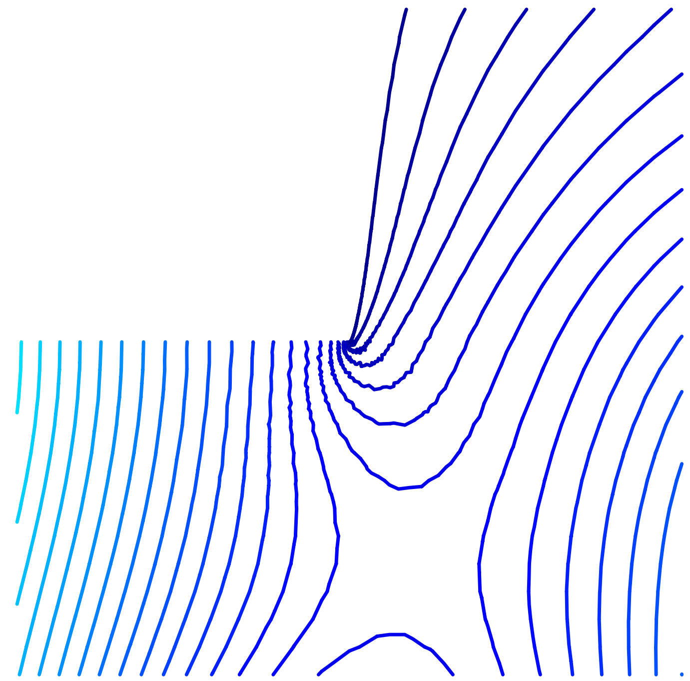
And E_{wr} is wrong, because for non-convex polyhedra $H_N \neq X_N$.

The standard nodal “approximation” of the first singularity

The exact solution $\left(\mathrm{grad}(r^{\frac{2}{3}}\sin\frac{2\theta}{3})\right)_1$.



Computation with $\mathbb{Q}_3$ **elements.**



Non- H^1 **singularities are** ***not approximated*****.**

Variational singularities derive from potentials

Any $\operatorname{grad}\varphi$ with $\varphi \in D(\Delta^{\mathrm{Dir}})$ belongs to X_N, where

$$D(\Delta^{\mathrm{Dir}}) = \{\varphi \in H_0^1(\Omega)\ ;\quad \Delta\varphi \in L^2(\Omega)\}.$$

For non-convex polyhedra, $\exists K_{\mathrm{Dir}} \neq \{0\}$ such that

$$\left(H^2 \cap H_0^1(\Omega)\right) \oplus K_{\mathrm{Dir}} = D(\Delta^{\mathrm{Dir}}).$$

There holds

$$\boxed{X_N = H_N \oplus \operatorname{grad}\left(K_{\mathrm{Dir}}\right)}$$

3D : For non-convex polyhedra, $\dim K_{\mathrm{Dir}} = +\infty$

2D : For non-convex polygons, $\dim K_{\mathrm{Dir}} = \#$ *nonconvex corners*

Boundary penalization

Boundary-penalized bilinear form $a[s,\lambda]$ **defined on** W **as**

$$\int_\Omega \operatorname{rot} E\cdot \operatorname{rot}\widetilde{E} + s\operatorname{div} E\ \operatorname{div}\widetilde{E} + \lambda\int_{\partial\Omega}(E\times n)\cdot(\widetilde{E}\times n)\ ,$$

with the variational space

$$W = \{E\in H(\mathrm{rot})\cap H(\mathrm{div})\ ;\quad E\times n|_{\partial\Omega}\in L^2(\partial\Omega)\}\,.$$

Theorem [CoDa'98] : Ω *Lipschitz domain:* $\mathcal{C}^\infty(\overline{\Omega})^3$ *is dense in* W **.**

Spurious eigenfrequencies (the same): $\omega[s,\lambda]^2 = s\nu$ **with Dirichlet eigenvalues** ν **.**

The solutions $\omega[s,\lambda]$ **independent of** s **tend to the Maxwell e-frequencies**

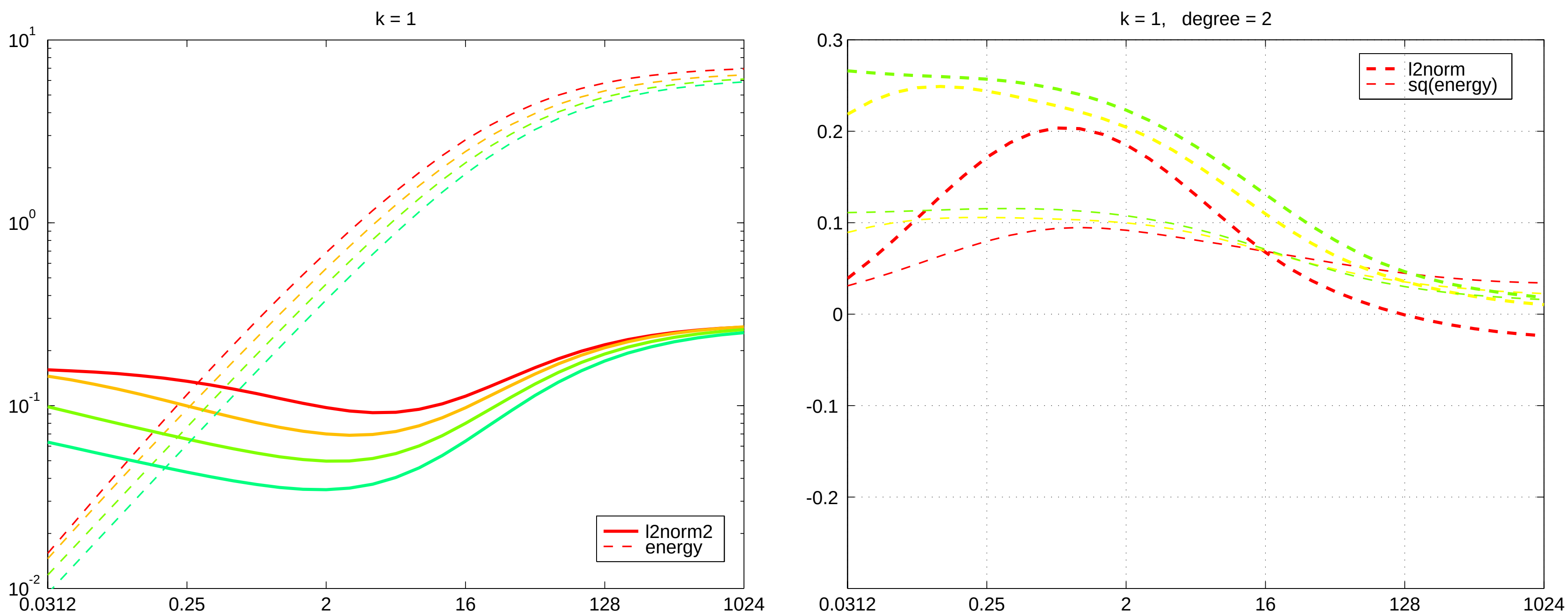
$$\big|\omega[s,\lambda]-\omega[s]\big| = \mathcal{O}(\tfrac{1}{\lambda}) \quad \text{as } \lambda\to\infty$$

However, convergence rates. . . are not known, and practical convergence is very poor.

Boundary penalization : non promising results

Comput. of 1^{st} singularity with $\mathbb{P}_2$ elements in 4 nested regular triangular meshes.

Quadratic errors and convergence rates for $s = 1$ and λ from 2^{-5} to 2^{10}.



Weighted regularization : The idea

Regularize the divergence via a term $s\langle \operatorname{div} E, \operatorname{div} \widetilde{E} \rangle_Y$ **with an intermediate space**

$$L^2(\Omega) \subset Y \subset H^{-1}(\Omega).$$

The variational space is then

$$X_N[Y] = \{E \in H_0(\mathrm{rot}) \mid \quad \operatorname{div} E \in Y\}.$$

We take Y **as a weighted** L^2 **space:**

$$\langle \operatorname{div} E, \operatorname{div} \widetilde{E} \rangle_Y = \int_\Omega \sigma \ \operatorname{div} E \ \operatorname{div} \widetilde{E} \, \mathrm{d}x$$

where

$$\sigma(x) = d(x)^\alpha \quad \text{with} \quad 0 \le \alpha \le 2$$

with $d(x) = \operatorname{dist}(x, \mathfrak{S})$ **the distance function to the set** $\mathfrak{S}$ **of**

non-convex corners for a polygon and non-convex edges for a polyhedron.

Weighted penalization : A density result

Define the Laplace-Dirichlet operator $\Delta^{\mathrm{Dir}}[Y]$ as

$$\Delta^{\mathrm{Dir}}[Y]: \quad \mathcal{D}(\Delta^{\mathrm{Dir}}[Y]) := \left\{\varphi \in H_0^1(\Omega) \mid \Delta\varphi \in Y\right\} \longrightarrow Y$$
$$\varphi \longmapsto \Delta\varphi.$$

Theorem [CoDa'00] :

(i) *Any element $u \in X_N[Y]$ can be decomposed into the sum*

$$u = w + \operatorname{grad}\varphi, \quad \textit{with} \quad w \in H_N \quad \textit{and} \quad \varphi \in \mathcal{D}(\Delta^{\mathrm{Dir}}[Y]).$$

(ii) *If $H^2 \cap H_0^1(\Omega)$ is dense in $\mathcal{D}(\Delta^{\mathrm{Dir}}[Y])$ for the graph norm, then H_N is dense in $X_N[Y]$.*

(iii) *$H^2 \cap H_0^1(\Omega)$ is closed in $\mathcal{D}(\Delta^{\mathrm{Dir}}[Y])$ if and only if H_N is closed in $X_N[Y]$.*

(iv) *$\mathcal{D}(\Delta^{\mathrm{Dir}}[Y])$ is contained in $H^2(\Omega)$ if and only if $X_N[Y] = H_N$.*

How to choose the weight for a polyhedron

Edges e, opening angles ω_e.
Corners c, Laplace Dirichlet singularity exponents λ_c^{Dir}.

With $d(x) = \mathrm{dist}(x, \mathfrak{S})$ the distance to the set $\mathfrak{S}$ of non-convex edges, the weight σ is defined as (we set $\alpha = 2\gamma$)

$$\sigma(x) = d(x)^{2\gamma} \quad \text{with} \quad 0 \leq \gamma \leq 1$$

When γ is close enough to 1, the domain $\mathcal{D}(\Delta^{\mathrm{Dir}}[Y])$ is a weighted space $V_\gamma^2(\Omega)$ of KONDRAT'EV type and the smooth functions are dense :

Theorem [CoDa'00] :

If $$\max_{e,c}\{1 - \frac{\pi}{\omega_e}\ ,\ \frac{1}{2} - \lambda_c^{\mathrm{Dir}}\} < \gamma \leq 1$$

then H_N *is dense in* $X_N[Y]$

$\Longrightarrow$ *Any Galerkin method converges.*

Approximation properties of the FEM spaces

The Maxwell eigenvectors E admit the following splitting

$$E = w + \operatorname{grad}\varphi, \quad \text{with} \quad w \in H_N \quad \text{and} \quad \varphi \in \mathcal{D}(\Delta^{\mathrm{Dir}}[Y]) .$$

In fact, w belongs to a space $H_N^{\mathrm{Neu}} \subset H_N$ where each component has the same regularity as solutions of the Neumann problem for Δ with a L^2 rhs and φ belongs to a weighted space K^∞_{Dir} determined by the only Dirichlet singularity exponents.

Assumptions on the FEM spaces $\mathfrak{X}_N^h$: $\exists \tau > 0$

$(\mathfrak{A}_1)$ $\forall w \in H_N^{\mathrm{Neu}}, \quad \inf_{w_h \in \mathfrak{X}_N^h} \|w - w_h\|_{H^1} \leq C\, h^\tau$

$(\mathfrak{A}_2)$ $\exists \Phi_h$: $\operatorname{grad} \Phi_h \subset \mathfrak{X}_N^h$, so that $\forall \varphi \in K^\infty_{\mathrm{Dir}}, \ \inf_{\varphi_h \in \Phi_h} \|\varphi - \varphi_h\|_{V^2_\gamma} \leq C\, h^\tau$

Theorem [CoDa'01] : *Estimate between e-vector E and e-vector $E_h \in \mathfrak{X}_N^h$:*

$$\|E - E_h\|_{X_N[Y]} \leq C(E)\, h^\tau$$

Convergence rates in 2D

The FEM spaces $\mathfrak{X}_N^h$ of nodal elements originating from

- $\mathbb{Q}_q$ rectangles for $q \geq 3$,
- $\mathbb{P}_q$ triangles for $q \geq 4$ ($q \geq 2$ on some triangulations),

on a h-uniform mesh, satisfy Assumptions $(\mathfrak{A}_1)$ – $(\mathfrak{A}_2)$ as soon as γ satisfies

$$\max_e \{1 - \tfrac{\pi}{\omega_e}\} < \gamma \leq 1$$

for any τ such that

$$\tau < \min\left\{\frac{\pi}{\omega_0} - 1 + \gamma\,,\ \frac{\pi}{\omega_1} - 1\right\}$$

where ω_0 is the largest non-convex angle and ω_1 the largest convex angle.

For the L-shaped domain, we obtain

$\tau < \gamma - \frac{1}{3}$, and for the optimal value $\gamma = 1$, $\tau < \frac{2}{3}$.

Illustration: Maxwell eigenvalues in a polygon

E-modes $\big(\Lambda[s], u[s]\big)$ **of the parameter-dependent problems associated with** $a[s]$

$$s \geq 0 \;:\quad a[s](u,v) \;=\; \int_\Omega \big(\operatorname{rot} u \;\operatorname{rot} v + s\, r^\alpha \;\operatorname{div} u \;\operatorname{div} v\big)\,\mathrm{d}x.$$

Here $\operatorname{rot} u = \partial_1 u_2 - \partial_2 u_1$ **and** $\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2$ **.**

The e-modes $\big(\Lambda[s], u[s]\big)$ **in** X_N **can be organized in 2 types**

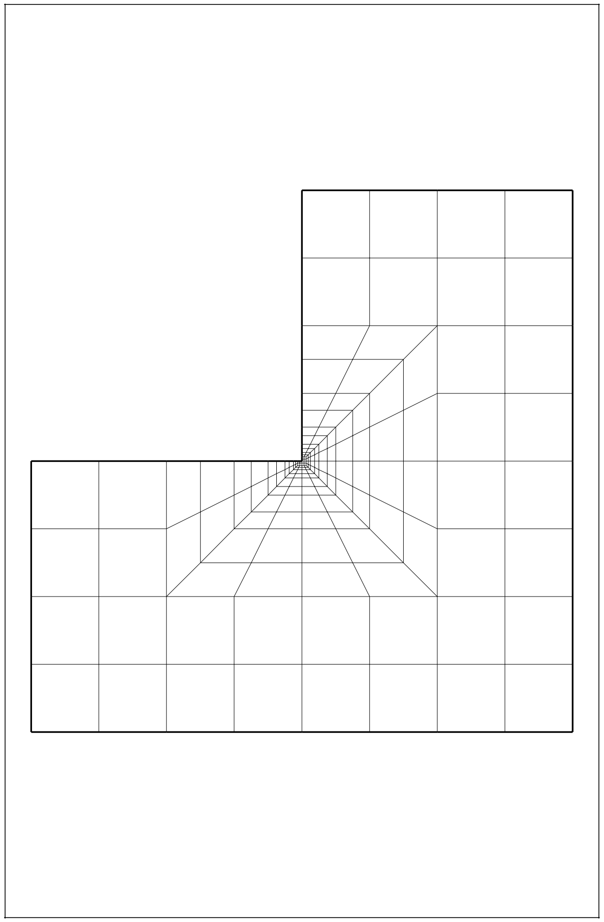
1) $\Lambda[s]$ ***independent*** **of** s **: they are the Maxwell e-values.**
$\big(\Lambda[s], u[s]\big) = \big(\nu, \overrightarrow{\operatorname{rot}}\varphi\big)$ **with the Neumann e-modes** $\big(\nu, \varphi\big)$ **of** $-\Delta$ **.**

2) $\Lambda[s]$ ***linear dependent*** **on** s **: they are the spurious e-values.**
$\big(\Lambda[s], u[s]\big) = \big(s\nu, \overrightarrow{\operatorname{grad}}\varphi\big)$ **with the Dirichlet e-modes** $\big(\nu, \varphi\big)$ **of the operator**

$$-r^{\alpha/2}\Delta\, r^{\alpha/2}.$$

Computations

Ω is the L-shape domain : $[0.5,1]\times[0.5,1]\setminus[0.75,1]\times[0.75,1]$.



Computation of the first 15 eigenvalues and eigenvectors $\Lambda[s]$ for

$$s = [2:100],$$
$$\alpha = 0,1,2,$$
$$q = 1,2,3,4$$

with the FEM library MÉLINA by D. MARTIN.

The computed eigenvalues are sorted depending on whether

$$\frac{\|\operatorname{rot} u\|^2}{s\|r^{\alpha/2}\operatorname{div} u\|^2}$$

is $\geq \rho$ or $\leq \rho^{-1}$ with a fixed $\rho \geq 1$.

Legend

For the next 8 figures.

In *Cyan*, the Laplace-Neumann eigenvalues, which coincide for 2D domains with Maxwell eigenvalues.

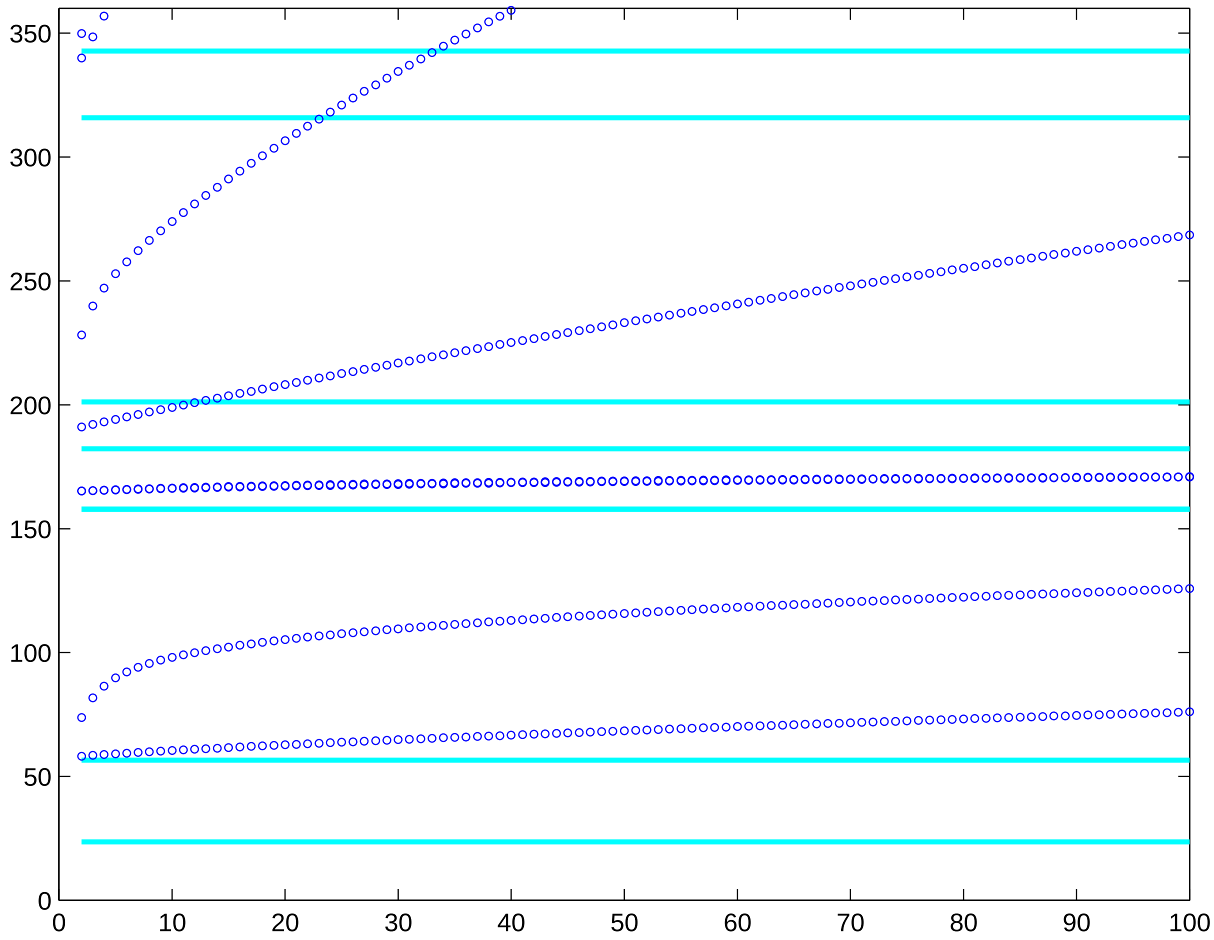
In *Blue*, the circles represent the computed $\mathbf{\Lambda}[s]$ with rot dominant eigenvectors.

In *Red*, the lines join the computed $\mathbf{\Lambda}[s]$ with div dominant eigenvectors.

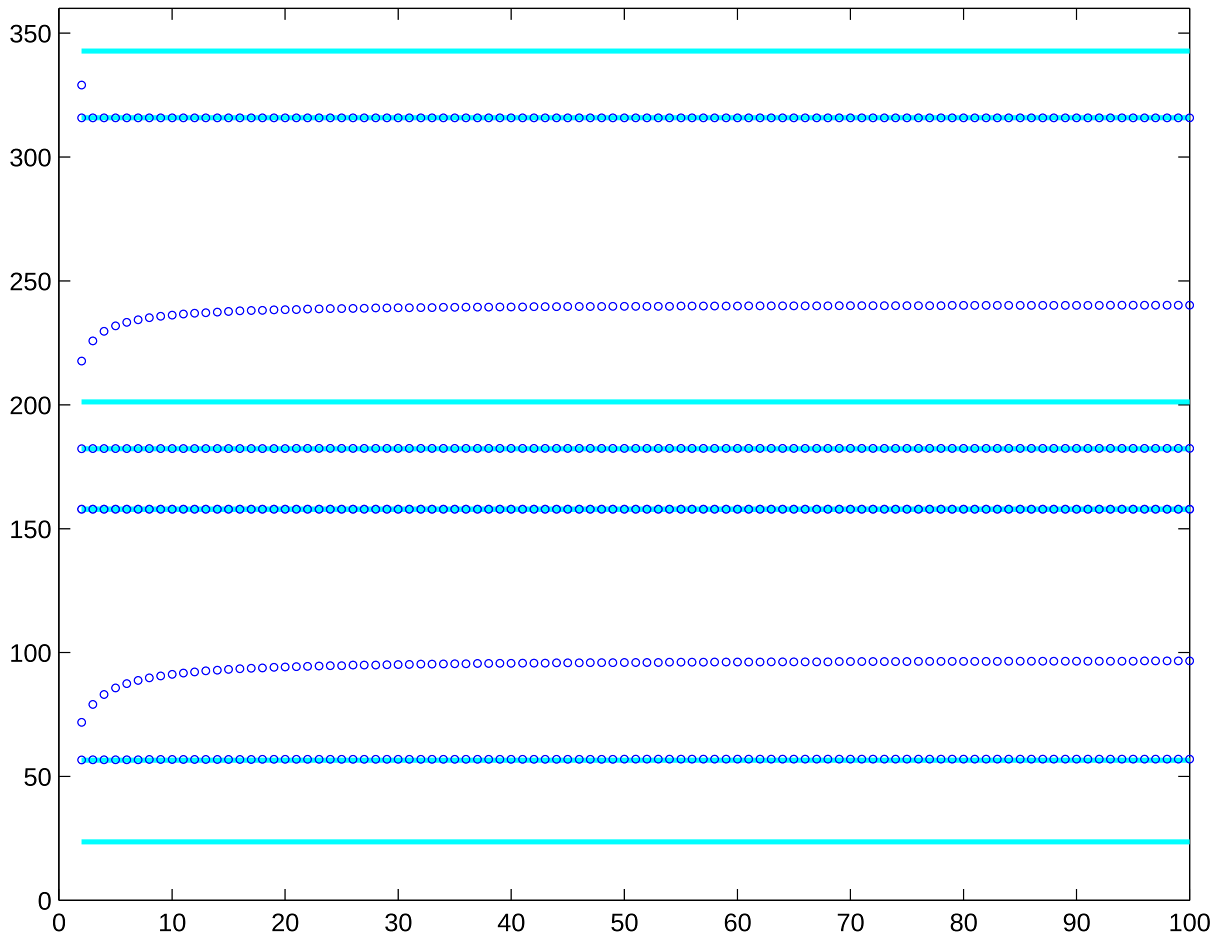
Abscissa : s

Ordinate : $\mathbf{\Lambda}[s]$.

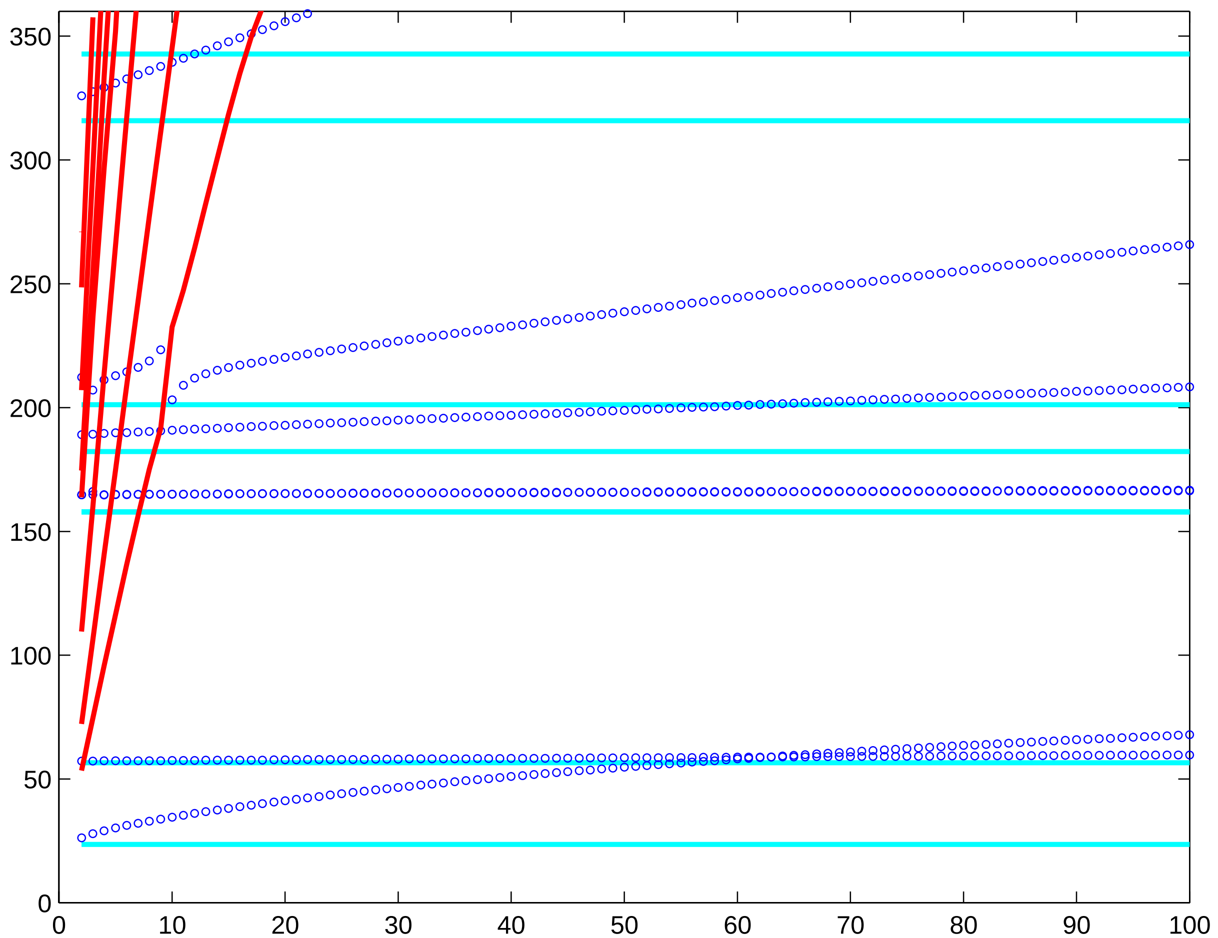
$q = 1\,,\ \alpha = 0$



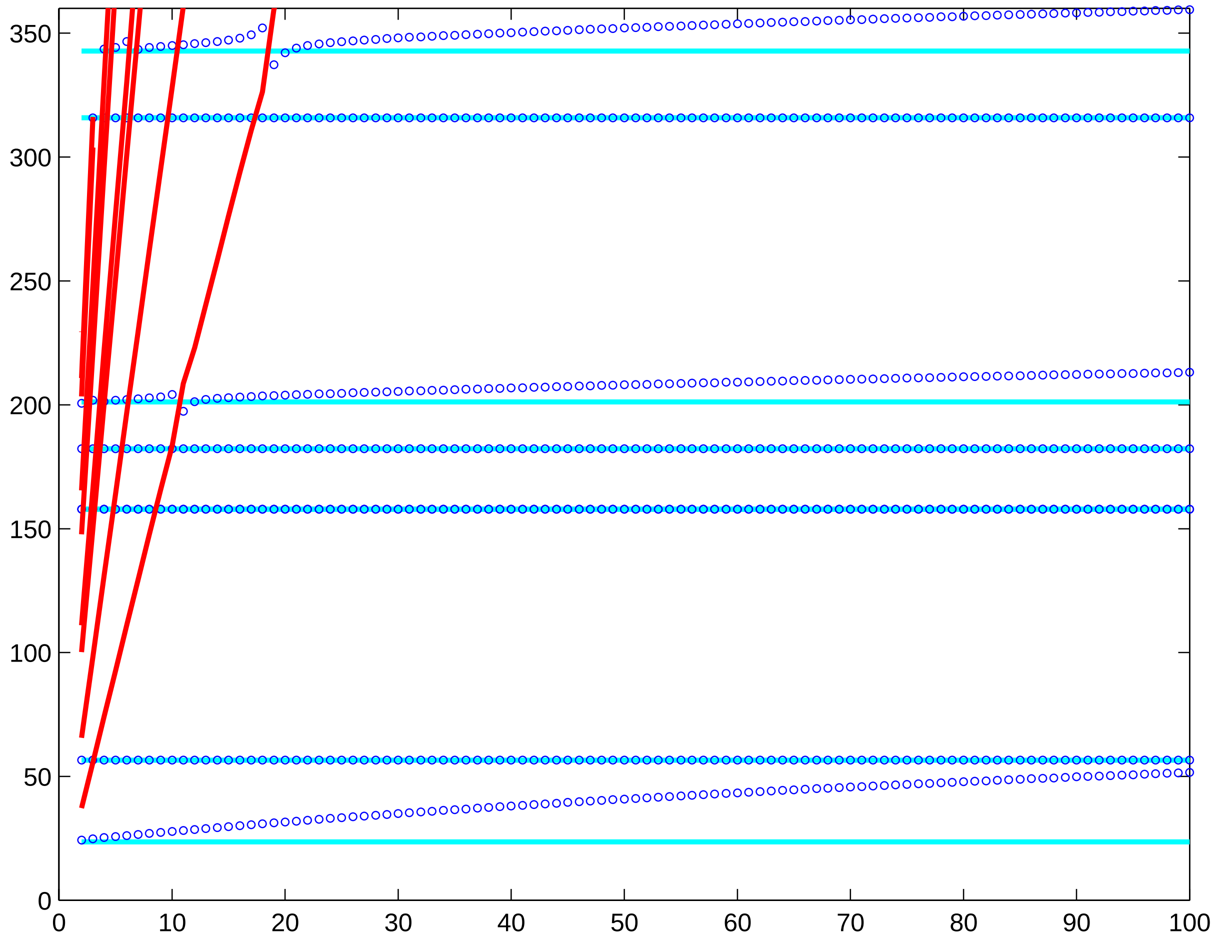
$q = 4, \ \alpha = 0$



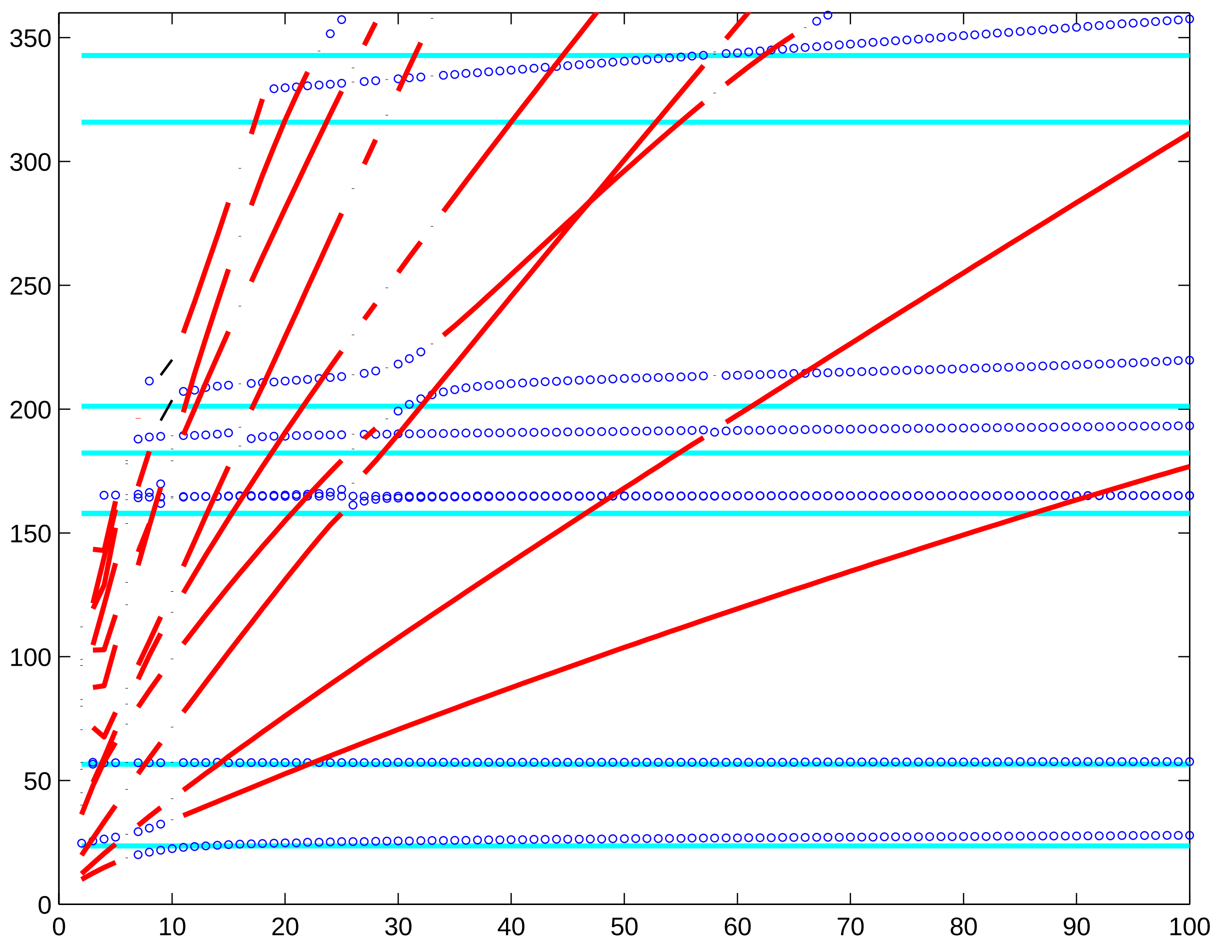
$q = 1, \ \alpha = 1$



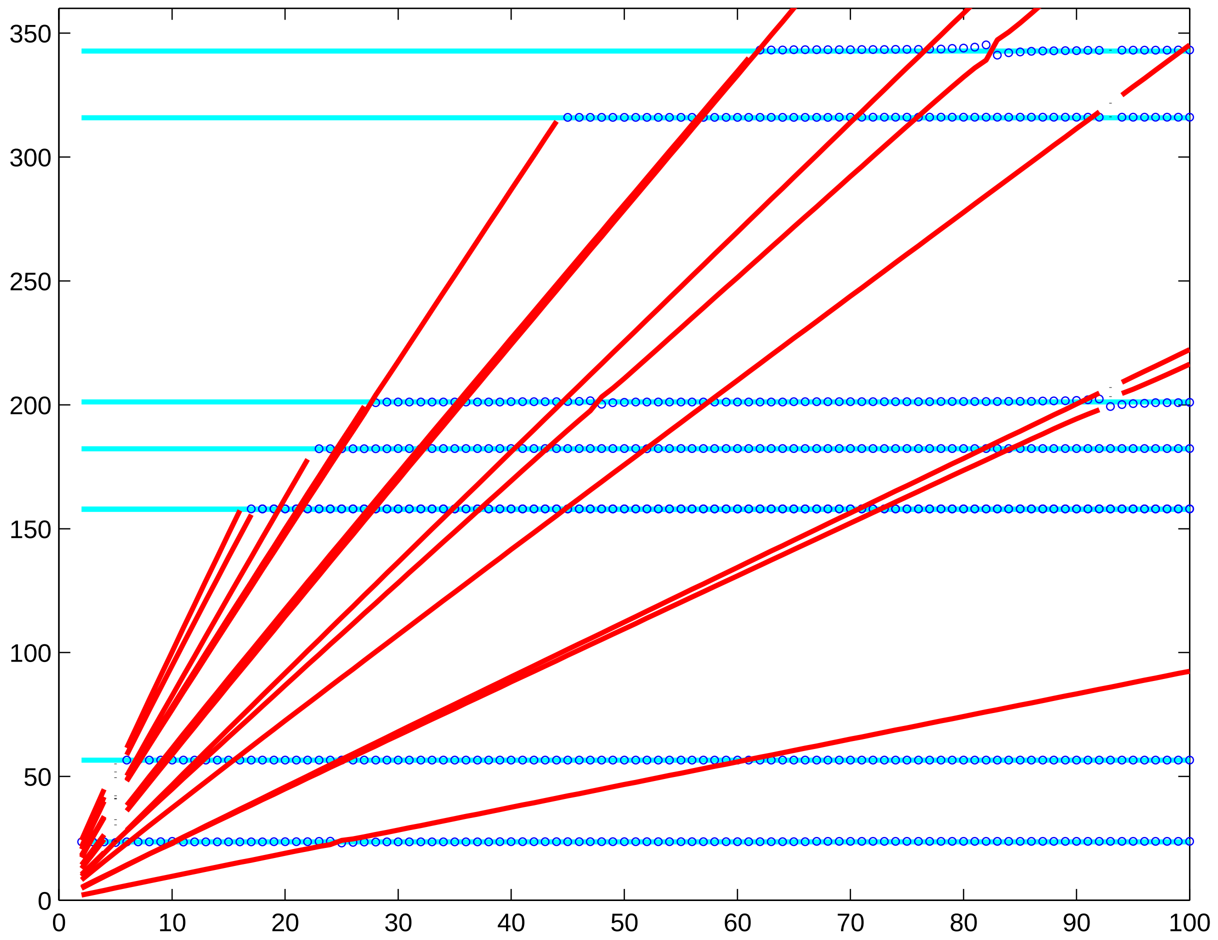
$$q = 4\,, \ \alpha = 1$$



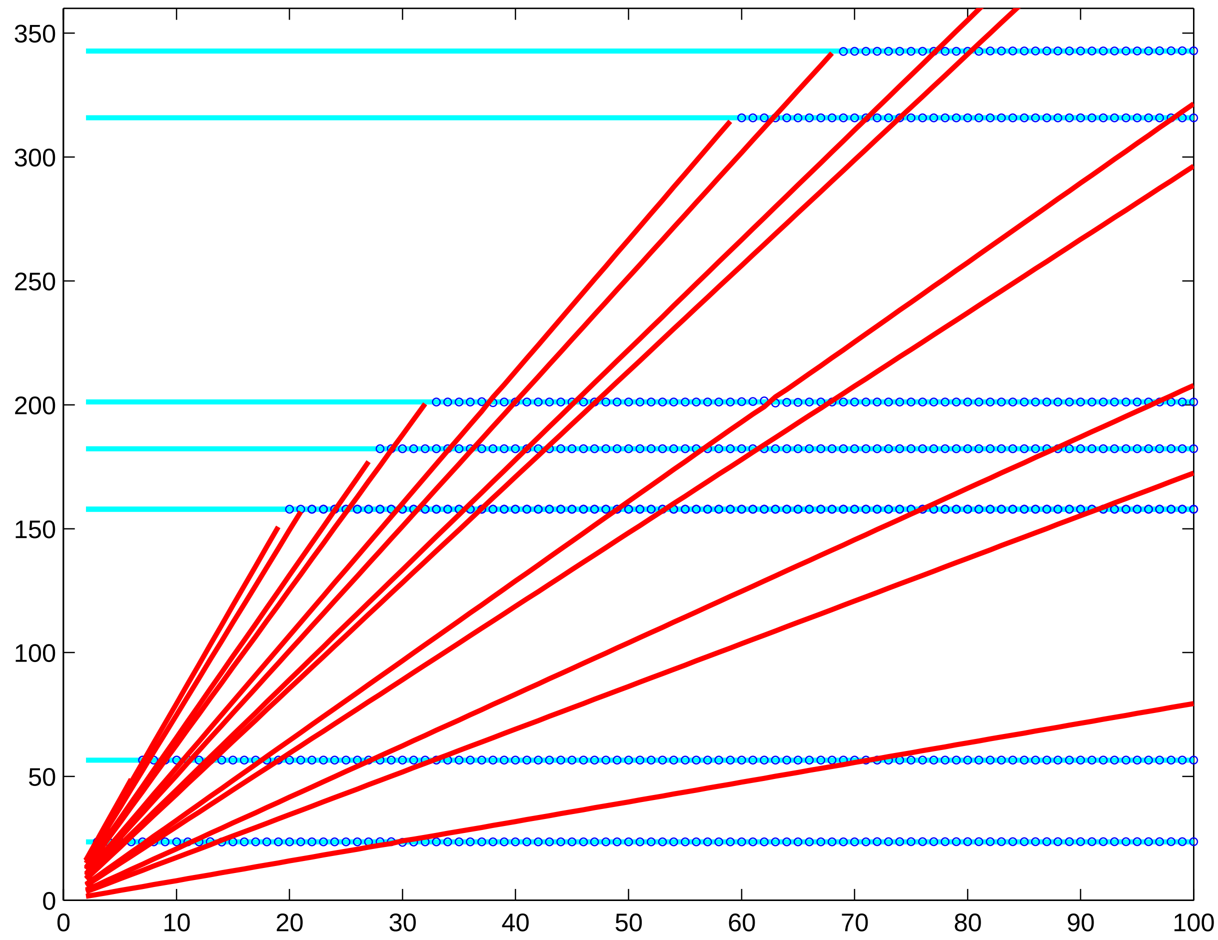
$q = 1, \ \alpha = 2$



$q = 2\,,\ \alpha = 2$



$q = 3\,,\ \alpha = 2$



$q = 4, \ \alpha = 2$