# Exponential Convergence of $hp$-FEM for Maxwell's Equations with Weighted Regularization in Polygonal Domains

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***Abstract.***

*The time-harmonic Maxwell equations do not have an elliptic nature by themselves. Their regularization by a divergence term is a standard tool to obtain equivalent elliptic problems. Nodal finite element discretizations of Maxwell's equations obtained from such a regularization converge to wrong solutions in any non-convex polygon. Modification of the regularization term consisting in the introduction of a weight restores the convergence of nodal FEM, providing optimal convergence rates for the $h$ Version of Finite Elements,* [21]. *We prove exponential convergence of $hp$ FEM for the weighted regularization of Maxwell's equations in plane polygonal domains provided the $hp$-FE spaces satisfy a series of axioms. We verify these axioms for several specific families of $hp$ finite element spaces.*

## Introduction

### 0.a FEM discretizations of Maxwell equations

When applied to the discretization of boundary value problems associated with standard elliptic equations such as Laplace's equation or the system of elasticity, the convergence of the Finite Element Method (FEM) is well understood by now, in particular for two-dimensional or three-dimensional domains with corners and edges. Low convergence rates caused by edge and vertex singularities can be overcome by a variety of techniques, such as isotropic or anisotropic algebraic mesh refinement ($h$ Version of FEM), increase of the polynomial degree (spectral methods or $p$ Version of FEM), or a combination of both, more precisely, by combining *geometric* mesh refinement with an increase of the polynomial degree $p$. This latter method is known as the $hp$ Version of FEM and was introduced by BABUŠKA et al. [4, 9, 10, 35]. We know from [5, 6, 27, 28] and [34] that, when the boundary of the domain and the data are piecewise analytic, the $hp$ Version of the FEM gives approximate solutions to elliptic problems with *exponential convergence rates*: This means that the error is divided by an asymptotically constant factor as the polynomial degree $p$ is increased by $1$, whereas the number $N$ of degrees of freedom is bounded by a power of $p$ (namely, $p^3$ in 2D). In two dimensions, the error has the order $e^{-bp} \simeq e^{-b\sqrt[3]{N}}$ with a positive constant $b$.

Time-harmonic Maxwell equations form a system of order $1$ and, by themselves, do not exhibit standard ellipticity. There are two main strategies to discretize them by FEM, see the survey papers [30, 22]. The first one enforces the divergence-free constraint with the help of a Lagrange multiplier and requires the use of special compatible polynomial bases and interpolants, respecting the commuting diagram properties (NÉDÉLEC and RAVIART-THOMAS elements, known as edge elements). The second strategy transforms the Maxwell system into an elliptic system of Helmholtz equations by "regularization", which consists of adding in the variational formulation a divergence term $(\mathbf{u},\mathbf{v}) \mapsto \langle \operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}\rangle$ to the usual curl term $(\mathbf{u},\mathbf{v}) \mapsto \langle \mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}\rangle$. The new bilinear form is coercive on the space $\mathsf{X}_N$ of electric fields $\mathbf{u}$ with square integrable curl and divergence, satisfying the perfect conductor boundary condition $\mathbf{u}\times\mathbf{n} = 0$ on the boundary of the domain. Thus the discretization by a finite element method based on nodal elements appears promising, and of simpler use and analysis than the edge elements.

In practice, nodal discretizations of the Maxwell equations are suitable only for regular domains or at best for convex polygons or polyhedra. Indeed, if the domain has reentrant corners or edges, the subspace of $\mathsf{H}^1$ fields in $\mathsf{X}_N$ is closed in $\mathsf{X}_N$, without being dense, see [18, 24, 20]. Since any discrete conforming space based on a standard nodal finite element method is contained in $\mathsf{H}^1$, nodal FEM converges in this situation in general to a wrong solution, see [19].

Nevertheless, a slight modification of that method restores its full efficiency and accuracy: In [21], COSTABEL-DAUGE introduced a positive weight in the divergence term which does not alter the equivalence properties with the original Maxwell problem, but enlarges the associated energy space. They proved that there exist weight functions so that the subspace of continuous functions is now dense in the enlarged energy space restoring the possibility of Galerkin discretizations in electromagnetics based on nodal finite elements. In [21] it was also demonstrated that nodal $h$ Version FEM converge with optimal rates in the weighted energy space.

Numerical experiments [23] for the source problem as well as for the eigenvalue problem were first performed with the FE library MÉLINA [32], combining geometric mesh refinement towards the corners with simultaneous increase of the polynomial degree of the approximation. These numerical experiments showed exponential convergence well-known for $hp$-FEM applied to standard scalar second order elliptic equations. The experiments [23] were corroborated by computations with the $hp$-FE library CONCEPTS [26] using conforming $hp$-FEM on geometric meshes of quadrilaterals with hanging nodes. This raised hope that the exponential convergence of nodal $hp$-FEM for scalar problems could be transferred to Maxwell equations via weighted regularization.

The main task of the present paper is to prove that this is true: For a wide range of continuous, nodal $hp$ finite element families $\big(\mathfrak{M}_p, \mathfrak{X}_p\big)_{p\geq 1}$ based on geometric meshes $\mathfrak{M}_p$ and local polynomial approximation spaces $\mathfrak{X}_p$, we prove exponential convergence rates of the Galerkin approximations $\mathbf{u}_p$ to the solution $\mathbf{u}$ in the weighted energy norms.

The reason why judicious combinations of polynomial degrees and geometric mesh refinement give exponential convergence rates is the same as for the standard elliptic operators investigated in [5, 6]: The asymptotics of the solution at a corner is a linear combination of terms of the form $r^\alpha\psi(\theta)$. But, whereas for “standard” problems investigated in [5, 6] the exponent $\alpha$ is always $> 0$, for Maxwell problems in non-convex domains, $\alpha$ is $< 0$ (but still $> -1$) at any reentrant corner. The weight which we use in the regularization is then $r^\gamma$ with $-\alpha < \gamma \leq 1$. The structure of the weight is thus similar to that of the singularities and combines perfectly with the fundamental properties of $hp$-FEM.

One of the main difficulties with nodal FEM for Maxwell's equations is the strong singularity of the solutions. It is known that the most singular part (the non-$\mathsf{H}^1$ contribution corresponding to negative exponents $\alpha$) can be written in the form of the gradient of a singular potential: For finite regularity, this is the BIRMAN-SOLOMYAK decomposition ([12], see also [21, 17] in weighted spaces). Our exponential convergence proof for analytic data relies on generalizing the Birman-Solomyak decomposition to weighted analytic spaces.

Our $hp$-FEM is based on a coercive formulation in spaces for which the embedding into $\mathsf{L}^2$ is compact. Therefore, thanks to standard tools (Céa Lemma), our approximation results and the analytic regularity yield exponential convergence of Maxwell solutions at any fixed frequency. Moreover, as a direct consequence of the classical estimates of [8], we can derive also exponential convergence of $hp$-FEM approximations to Maxwell eigenvalues and eigenvectors.

This is in contrast to the situation with edge elements, where approximation estimates have to be combined with the proof of the *discrete compactness property* which is not obvious for the $p$ Version [14, 13]. The price to pay for circumventing the discrete compactness in our analysis is the construction of a $\mathscr{C}^1$ $hp$-interpolant. We emphasize that is merely a technicality of our proof for a discrete analog of the Birman-Solomyak decomposition, but has no influence on the $hp$-FE discretization which only uses nodal, Lagrangian $\mathscr{C}^0$ interpolants.

The $hp$ Version FEM for edge elements is now widely used in practice, see [1, 33] for example. It has not yet been thoroughly analyzed from a theoretical point of view for the Maxwell equations, however. A step in that direction is [2] where exponential convergence is proved for Raviart-Thomas elements when approximating a scalar Laplace equation in mixed form. If combined with our result on an “analytic Birman-Solomyak decomposition”, the approximation result of [2] can provide exponential convergence towards Maxwell solutions in the coercive case (e.g. in the presence of a non-zero conductivity).

### 0.b Plan and scope of the paper

We will concentrate on the following model situation: The domain $\Omega$ is a not necessarily convex polygon with corners $\boldsymbol{c}$ and openings $\omega_{\boldsymbol{c}} < 2\pi$. The Maxwell source problem consists in finding $\mathbf{u} \in \mathsf{L}^2(\Omega)^2$ with $\operatorname{curl}\mathbf{u} \in \mathsf{L}^2(\Omega)$, $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{u}\times\mathbf{n} = 0$ on $\partial\Omega$ such that $\mathbf{curl}\operatorname{curl}\mathbf{u} = \mathbf{f}$ where $\mathbf{f}$ is a divergence free field with analytic regularity. We postpone the analysis of the $hp$ FEM in three dimensions — the basic functional results of weighted regularization leading to the convergence of the $h$ Version are proved for three-dimensional polyhedra in [21].

In Section 1, we give a brief account of the weighted regularization introduced in [21]. Next, in Section 2, we study analytic regularity for our Maxwell boundary value problem on polygons. The main result in Theorem 2.7 and Corollary 2.8 gives a decomposition of the solution into a “regular” part and a gradient containing the main corner singularity. The regularity of both the regular part and the potential of the gradient are characterized in terms of weighted analytic spaces.

The subsequent part of the paper is devoted to $hp$ finite element convergence analysis. It is divided into an abstract part comprising Sections 3, 4, 5 and a specific part with applications in Section 6. The abstract part axiomatizes mesh and degree selection principles sufficient for exponential convergence for the specific examples of $hp$-FE spaces that we have in mind. These examples include the main classes of finite elements most frequently used in $hp$ methods:

- **(a)** Rectangles with hanging nodes, and $\mathbb{Q}^p$ polynomials,
- **(b)** Conforming parallelograms and triangles, using $\mathbb{Q}^p$ and $\mathbb{P}^p$ polynomials respectively,
- **(c)** Non-affine $\mathbb{Q}^1$ quadrilaterals with mapped $\mathbb{Q}^p$ polynomials.

Verification of the abstract axioms for these specific examples is done in Section 6.

The unified treatment of these (and other) examples requires a certain degree of generality in the hypotheses of the abstract part of our error analysis: we cannot stay within the framework of "affine families of finite elements" where the polynomial spaces on the elements are generated from one polynomial space on the reference element. For the non-affine quadrilaterals, approximation spaces on an element are generated from polynomial spaces on the reference element that are proper subspaces of $\mathbb{Q}^p$ and depend on the element. We do not, however, try to present a framework that is more abstract and general than strictly necessary.

In Section 3, we introduce the axioms to be satisfied by the families of meshes, and in Section 4, those relating to the elementwise spaces and interpolation operators. At each level, global exponential estimates are derived from generic local estimates, if applied to functions in suitable weighted analytic spaces. In Section 5, the axioms on the families of discrete spaces for the weighted regularization are introduced and the main convergence result (Theorem 5.2) is immediately derived. In Section 6, we exhibit the different interpolation operators corresponding to concrete situations **(a)**, **(b)**, and **(c)**. The proofs of the local estimates rely on more technical results (some of them "classical"), which we have gathered in the appendix Section 8. We draw some conclusions in Section 7.

In this work, we assume for simplicity that the polynomial degree $p$ of the elements is constant throughout the geometric mesh $\mathfrak{M}_p$. We point out, however, that all our proofs and results carry over to the case of linearly increasing polynomial degree vectors with positive slope (see e.g. [34]). Our analysis simplifies even more in the last layer around the corners, where our interpolant vanishes identically, thereby avoiding the analysis of low order interpolants in weighted spaces in these elements.

The abstract $hp$ convergence framework presented in this paper simplifies the proof of exponential convergence also in other situations because it is split into different estimates which are proved independently, inside separate modules. For the more interesting and difficult case of three-dimensional polyhedra, it can serve as a strategy for the convergence analysis of the $hp$-FEM. The main difficulty that will have to be overcome in the 3-d case is the precise description of the analytic regularity of solutions of "standard" elliptic problems as well as Maxwell's equations on polyhedral domains. This analytic regularity is available for 2-d problems, but has only partially been analyzed for 3-d problems [27, 28]. Another difficulty in the 3-d case are anisotropic estimates (see [3, 16]) that are needed when mesh refinements lead to strongly anisotropic meshes. In two dimensions, we can exclude strong anisotropy and stay in the framework of shape-regular elements.

## 1 Weighted regularization

The domain $\Omega$ is a Lipschitz polygonal domain in $\mathbb{R}^2$ and the Cartesian coordinates are $x = (x_1, x_2)$. Let $\mathsf{H}_0(\mathrm{curl}\,;\Omega)$ be the subspace of $\mathsf{L}^2$ fields $\mathbf{u} = (u_1, u_2)$ in $\Omega$ such that $\mathrm{curl}\,\mathbf{u} \in \mathsf{L}^2(\Omega)$ (with $\mathrm{curl}\,\mathbf{u} = \partial_1 u_2 - \partial_2 u_1$) and $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$ (with $\mathbf{n}$ the unit outward normal field to $\partial\Omega$). The source problem reads: given $\mathbf{f} \in \mathbf{L}^2(\Omega) := \mathsf{L}^2(\Omega)^2$ with $\mathrm{div}\,\mathbf{f} = 0$,

$$\text{find } \mathbf{u} \in \mathsf{H}_0(\mathrm{curl}\,;\Omega) \text{ with } \mathrm{div}\,\mathbf{u} = 0 : \qquad \forall \mathbf{v} \in \mathsf{H}_0(\mathrm{curl}\,;\Omega), \quad \int_\Omega \mathrm{curl}\,\mathbf{u}\ \mathrm{curl}\,\mathbf{v}\ \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v}\ \mathrm{d}x. \tag{1.1}$$

Let $\mathsf{X}_N(\Omega)$ be the subspace of $\mathsf{H}_0(\mathrm{curl}\,;\Omega)$:

$$\mathsf{X}_N(\Omega) := \big\{ \mathbf{u} \in \mathsf{H}_0(\mathbf{curl}\,;\Omega) \mid \quad \mathrm{div}\,\mathbf{u} \in \mathsf{L}^2(\Omega) \big\}.$$

Then $\mathbf{u}$ solves (1.1) if and only if $\mathbf{u}$ solves

$$\text{Find } \mathbf{u} \in \mathsf{X}_N(\Omega) : \qquad \forall \mathbf{v} \in \mathsf{X}_N(\Omega), \quad \int_\Omega \mathrm{curl}\,\mathbf{u}\ \mathrm{curl}\,\mathbf{v} + \mathrm{div}\,\mathbf{u}\ \mathrm{div}\,\mathbf{v}\ \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v}\ \mathrm{d}x. \tag{1.2}$$

The variational formulation (1.2) allows to prove the existence and uniqueness of solution and, moreover, to determine the singularities of $\mathbf{u}$ near the corners of $\Omega$, see [20].

Let $\mathscr{C}$ be the set of the corners $\boldsymbol{c}$ of $\Omega$ and $r_{\boldsymbol{c}}$ the distance function to $\boldsymbol{c}$. Let $\omega_{\boldsymbol{c}}$ denote the interior opening angle of $\Omega$ at vertex $\boldsymbol{c}$. Let $\boldsymbol{\gamma} = (\gamma_{\boldsymbol{c}})_{\boldsymbol{c}\in\mathscr{C}}$ be a multi-exponent and denote by $\underline{r}^{\boldsymbol{\gamma}}$ the weight function

$$\underline{r}^{\boldsymbol{\gamma}} = \min_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\gamma_c}(x).$$

The regularization with weight consists in introducing $\underline{r}^{\boldsymbol{\gamma}}$ in the definition of the variational space and formulation: Let

$$\mathsf{X}_N^{\boldsymbol{\gamma}}(\Omega) := \big\{ \mathbf{u} \in \mathsf{H}_0(\mathbf{curl}\,;\Omega) \mid \quad \underline{r}^{\boldsymbol{\gamma}}\,\mathrm{div}\,\mathbf{u} \in \mathsf{L}^2(\Omega) \big\},$$

with its norm $\|\mathbf{u}\|_{\mathsf{X}_N^{\boldsymbol{\gamma}}(\Omega)} = \big( \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\,\mathrm{curl}\,\mathbf{u}\|^2_{\mathsf{L}^2(\Omega)} + \|r^{\boldsymbol{\gamma}}\,\mathrm{div}\,\mathbf{u}\|_{\mathsf{L}^2(\Omega)} \big)^{1/2}$.
The corresponding variational formulation is

$$\text{Find } \mathbf{u} \in \mathsf{X}_N^{\boldsymbol{\gamma}}(\Omega) : \qquad \forall \mathbf{v} \in \mathsf{X}_N^{\boldsymbol{\gamma}}(\Omega), \quad \int_\Omega \mathrm{curl}\,\mathbf{u}\ \mathrm{curl}\,\mathbf{v}\ +\ \underline{r}^{\boldsymbol{\gamma}}\,\mathrm{div}\,\mathbf{u}\ \underline{r}^{\boldsymbol{\gamma}}\,\mathrm{div}\,\mathbf{v}\ \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v}\ \mathrm{d}x. \tag{1.3}$$

From the $\mathsf{X}_N^{\boldsymbol{\gamma}}(\Omega)$-coercivity of the bilinear form, we get existence and uniqueness of a solution of (1.3), and there holds, [21]

**Theorem 1.1** (i) *For any multi-exponent* $\boldsymbol{\gamma} = (\gamma_{\boldsymbol{c}})_{\boldsymbol{c}\in\mathscr{C}}$ *with* $\gamma_{\boldsymbol{c}} \in [0,1]$, *the field* $\mathbf{u}$ *solves* (1.1) *if and only if* $\mathbf{u}$ *solves* (1.3).
(ii) *For any multi-exponent* $\boldsymbol{\gamma} = (\gamma_{\boldsymbol{c}})_{\boldsymbol{c}\in\mathscr{C}}$ *such that*

$$\forall \boldsymbol{c} \in \mathscr{C}, \quad 0 \le \gamma_{\boldsymbol{c}} \quad \textit{and} \quad 1 - \pi/\omega_{\boldsymbol{c}} < \gamma_{\boldsymbol{c}} \le 1, \tag{1.4}$$

*the space* $\mathsf{H}^1_N(\Omega)$ *of* $\mathsf{H}^1$ *fields with tangential boundary condition is* dense *in* $\mathsf{X}^{\gamma}_N(\Omega)$.

The finite element method for the weighted regularization consists in Galerkin approximation based on finite dimensional subspaces $\mathfrak{X}^p$ of $\mathsf{X}^{\gamma}_N(\Omega)$:

$$\text{Find } \mathbf{u}_p \in \mathfrak{X}^p : \quad \forall \mathbf{v}_p \in \mathfrak{X}^p, \quad \int_\Omega \operatorname{curl} \mathbf{u}_p \, \operatorname{curl} \mathbf{v}_p \, + \, \underline{r}^{\gamma} \operatorname{div} \mathbf{u}_p \, \underline{r}^{\gamma} \operatorname{div} \mathbf{v}_p \, \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v}_p \, \mathrm{d}x. \tag{1.5}$$

By Céa's lemma we have

$$\|\mathbf{u} - \mathbf{u}_p\|_{\mathsf{X}^{\gamma}_N(\Omega)} \le C \|\mathbf{u} - \mathbf{v}_p\|_{\mathsf{X}^{\gamma}_N(\Omega)}, \quad \forall \mathbf{v}_p \in \mathfrak{X}^p. \tag{1.6}$$

We are going to construct a class of families of finite element approximation spaces $\big(\mathfrak{X}^p\big)_{p\in\mathbb{N}}$ so that

- The dimension of $\mathfrak{X}^p$ is $\mathcal{O}\big(p^3\big)$,
- We have an error estimate $\|\mathbf{u} - \mathbf{u}_p\|_{\mathsf{X}^{\gamma}_N(\Omega)} \le Ce^{-bp}$ with $C,\ b > 0$ independent of $p$, provided the data $\mathbf{f}$ has certain analyticity properties.

## 2 Analytic regularity

The error analysis of our method is based on two principles:

1. The decomposition $\mathbf{u} = \mathbf{w} + \mathbf{grad}\,\varphi$ of the solution into a regular part and a gradient, in the style of BIRMAN-SOLOMYAK [11]. Note that this is not a Hodge or Helmholtz type decomposition where $\mathbf{u}$ is represented by means of a vector and a scalar potential. The latter would not provide the required additional regularity.

2. The use of weighted analytic function spaces of the type of Babuška-Guo's "countably normed spaces" [5].

In [21], the error analysis of the $h$ Version FEM was similarly based on the Birman-Solomyak decomposition and regularity in weighted Sobolev spaces of (arbitrarily high but) finite order.

As usual for Maxwell's equations, we will obtain our regularity results as corollaries of better known results for the Laplace operator.

## 2.a Corners

We gather in this section the notations relating to the geometry of the domain which will be used all over the paper. We recall that we denote by $\mathscr{C}$ the set of the corners $\boldsymbol{c}$ of $\Omega$ and by $\omega_{\boldsymbol{c}}$ the opening of $\Omega$ in $\boldsymbol{c}$. By $\mathscr{C}_\star$ we will denote the set of non-convex corners $\boldsymbol{c}$ of $\Omega$ for which $\omega_{\boldsymbol{c}} > \pi$. The set $\mathscr{C}_\star$ can be empty in the case of a convex polygon. In this case, the analysis will simplify, because we will not need the Birman-Solomyak decomposition. Geometric mesh refinement is generally needed also towards convex corners to achieve exponential convergence.

We also introduce an open covering $(\Theta_0, \Theta_{\boldsymbol{c}})$ of $\Omega$ separating the corners:

$$\Omega = \Theta_0 \cup \bigcup_{\boldsymbol{c}\in\mathscr{C}} \Theta_{\boldsymbol{c}} \;:\quad \big(\overline{\Theta}_{\boldsymbol{c}}\big)_{\boldsymbol{c}\in\mathscr{C}} \text{ mutually disjoint,} \quad \forall \boldsymbol{c}\in\mathscr{C},\;\; (\boldsymbol{c}\in\overline{\Theta}_{\boldsymbol{c}} \text{ and } \boldsymbol{c}\notin\overline{\Theta}_0). \tag{2.1}$$

We will further need a "larger" covering $(\Theta'_0, \Theta'_{\boldsymbol{c}})$ defined as follows: For any corner $\boldsymbol{c}$ let $\Theta'_{\boldsymbol{c}}$ be a neighborhood such that $\overline{\Theta}'_{\boldsymbol{c}}$ contains $\boldsymbol{c}$ and no other corner. We assume that $\Theta'_{\boldsymbol{c}}$ is larger than $\Theta_{\boldsymbol{c}}$, which means that there exist two neighborhoods $\mathscr{V}_{\boldsymbol{c}} \subset\subset \mathscr{V}'_{\boldsymbol{c}}$ of $\boldsymbol{c}$ in $\mathbb{R}^2$ such that $\Theta_{\boldsymbol{c}} = \mathscr{V}_{\boldsymbol{c}}\cap\Omega$ and $\Theta'_{\boldsymbol{c}} = \mathscr{V}'_{\boldsymbol{c}}\cap\Omega$. In a similar way, there exist open sets $\mathscr{V}_0 \subset\subset \mathscr{V}'_0$ disjoint from $\mathscr{C}$ such that $\Theta_0 = \mathscr{V}_0\cap\Omega$ and $\Theta'_0 = \mathscr{V}'_0\cap\Omega$.

Let $\Gamma_{\boldsymbol{c}}$ and $\Gamma_0$ denote the respective parts of the boundary of $\Omega$:

$$\Gamma_{\boldsymbol{c}} = \partial\Omega\cap\overline{\Theta}_{\boldsymbol{c}} \quad\text{and}\quad \Gamma_0 = \partial\Omega\cap\overline{\Theta}_0.$$

and let us define similarly $\Gamma'_{\boldsymbol{c}}$ and $\Gamma'_0$ relating to $\Theta'_{\boldsymbol{c}}$ and $\Theta'_0$.

## 2.b Spaces

We first recall some definitions of weighted spaces, *cf* [31]. Let $\boldsymbol{\beta} = (\beta_{\boldsymbol{c}}) \in \mathbb{R}^{|\mathscr{C}|}$ be a multi-exponent and $m$ a non-negative integer and let $(r_{\boldsymbol{c}}, \theta_{\boldsymbol{c}})$ be polar coordinates centered in $\boldsymbol{c}$.

For any $v\in\mathscr{D}'(\Omega)$ we define the semi-norm

$$|v|_{\mathsf{K}^m_{\boldsymbol{\beta}}(\Omega)} = \Big(|v|^2_{\mathsf{H}^m(\Theta_0)} + \sum_{\boldsymbol{c}\in\mathscr{C}}\sum_{|\alpha|=m} \|r_{\boldsymbol{c}}^{\beta_{\boldsymbol{c}}+|\alpha|}\partial^\alpha v\|^2_{\mathsf{L}^2(\Theta_{\boldsymbol{c}})}\Big)^{1/2}. \tag{2.2}$$

The weighted space $\mathsf{K}^m_{\boldsymbol{\beta}}(\Omega)$ is the space of $v\in\mathscr{D}'(\Omega)$ such that for all $k$, $0\le k\le m$, the semi-norm $|v|_{\mathsf{K}^m_{\boldsymbol{\beta}}(\Omega)}$ is finite.

Note that any derivative $\partial^\alpha$ is continuous from $\mathsf{K}^m_{\boldsymbol{\beta}-|\alpha|}(\Omega)$ into $\mathsf{K}^{m-|\alpha|}_{\boldsymbol{\beta}}(\Omega)$. Moreover $\mathsf{K}^m_{\boldsymbol{\beta}}(\Omega)$ is contained in $\mathsf{H}^m(\Omega)$ if and only if $\boldsymbol{\beta}\le -m$ (i.e. $\beta_{\boldsymbol{c}}\le -m$ for any corner $\boldsymbol{c}$).

We also need the corresponding trace spaces. For any $v\in\mathscr{D}(\mathbb{R}^2\setminus\mathscr{C})$ (i.e. with support outside the set of corners) we define the semi-norm

$$|v|_{\mathsf{K}^m_{\boldsymbol{\beta}}(\partial\Omega)} = \Big(|v|^2_{\mathsf{H}^m(\Gamma_0)} + \sum_{\boldsymbol{c}\in\mathscr{C}} \|r_{\boldsymbol{c}}^{\beta_{\boldsymbol{c}}+m}\partial^m_{r_{\boldsymbol{c}}} v\|^2_{\mathsf{L}^2(\Gamma_{\boldsymbol{c}})}\Big)^{1/2} \tag{2.3}$$

and the space $\mathsf{K}^m_{\boldsymbol{\beta}}(\partial\Omega)$ is the closure of $\mathscr{D}(\mathbb{R}^2\setminus\mathscr{C})$ for the norm $\big(\sum_{0\le k\le m}|v|^2_{\mathsf{K}^k_{\boldsymbol{\beta}}(\partial\Omega)}\big)^{1/2}$. Note that for positive non-integer $s$, $\mathsf{K}^s_{\boldsymbol{\beta}}(\partial\Omega)$ can be defined by interpolation and the trace operator is continuous from $\mathsf{K}^m_{\boldsymbol{\beta}-1/2}(\Omega)$ into $\mathsf{K}^{m-1/2}_{\boldsymbol{\beta}}(\partial\Omega)$.

The analytic weighted space $\mathsf{A}_{\boldsymbol\beta}(\Omega)$ is the space of $v \in \bigcap_{m\in\mathbb{N}} \mathsf{K}^m_{\boldsymbol\beta}(\Omega)$ such that

$$\exists C > 0, \quad \forall m \geq 0, \quad |v|_{\mathsf{K}^m_{\boldsymbol\beta}(\Omega)} \leq C^{m+1} m! \tag{2.4}$$

and the trace space $\mathsf{A}_{\boldsymbol\beta}(\partial\Omega)$ is the space of $v \in \bigcap_{m\in\mathbb{N}} \mathsf{K}^m_{\boldsymbol\beta}(\partial\Omega)$ such that $\exists C > 0$, $\forall m \geq 0$, $|v|_{\mathsf{K}^m_{\boldsymbol\beta}(\partial\Omega)} \leq C^{m+1} m!$.

Thus the derivative $\partial^\alpha$ is continuous from $\mathsf{A}_{\boldsymbol\beta-|\alpha|}(\Omega)$ into $\mathsf{A}_{\boldsymbol\beta}(\Omega)$ and the trace is continuous from $\mathsf{A}_{\boldsymbol\beta-1/2}(\Omega)$ into $\mathsf{A}_{\boldsymbol\beta}(\partial\Omega)$.

We will also use the localized version of these spaces in each neighborhood $\Theta_{\boldsymbol c}$: then we only need one weight $\beta_{\boldsymbol c}$ and define $\mathsf{K}^m_{\beta_{\boldsymbol c}}(\Theta_{\boldsymbol c})$, $\mathsf{A}_{\beta_{\boldsymbol c}}(\Theta_{\boldsymbol c})$ in the natural way.

The following result gives the analytic weighted regularity of corner singular functions:

**Lemma 2.1** *Let $\nu \in \mathbb{R}$ and $\psi$ an analytic function on $[0, \omega_{\boldsymbol c}]$. Then the function $r^\nu_{\boldsymbol c}\psi(\theta_{\boldsymbol c})$ belongs to $\mathsf{A}_{-1-\beta_{\boldsymbol c}}(\Theta_{\boldsymbol c})$ for all $\beta_{\boldsymbol c} < \nu$.*

The spaces $\mathsf{A}_{\boldsymbol\beta}(\Omega)$ are related to the spaces $\mathrm{B}^l_{\boldsymbol\beta}(\Omega)$ of BABUŠKA-GUO [5]. If $0 < \beta < 1$, $\mathsf{A}_{\boldsymbol\beta}(\Omega)$ coincides with $\mathrm{B}^0_{\boldsymbol\beta}(\Omega)$, whereas for $-1 < \beta < 0$, $\mathsf{A}_{\boldsymbol\beta}(\Omega)$ coincides with $\mathrm{B}^1_{\boldsymbol\beta+1}(\Omega)$. Finally, for $-2 < \beta < -1$, $\mathsf{A}_{\boldsymbol\beta}(\Omega)$ is a closed subspace of $\mathrm{B}^2_{\boldsymbol\beta+2}(\Omega)$ and differs from it by constants at the corner points.

### 2.c Shift theorem

Let $L$ be a properly elliptic $N \times N$ system of second order, homogeneous with constant coefficients. Let $B_1, \ldots, B_N$ be homogeneous boundary operators of orders $m_1, \ldots, m_N$ with constant coefficients on each edge of $\Omega$, satisfying the Shapiro-Lopatinski covering condition for $L$. Then, combining a dyadic partition of $\Theta'_{\boldsymbol c}$ and analytic type a priori estimates between pairs of nested annular domains together with an homogeneity argument, *cf* [15], we can prove (we use the notations of §2.a)

**Theorem 2.2** *Let $\mathbf{u} \in \mathsf{K}^2_{\beta_{\boldsymbol c}}(\Theta'_{\boldsymbol c})^N$ satisfy $L\mathbf{u} \in \mathsf{A}_{\beta_{\boldsymbol c}+2}(\Theta'_{\boldsymbol c})^N$ and $B_k\mathbf{u} \in \mathsf{A}_{\beta_{\boldsymbol c}+m_k+1/2}(\Gamma'_{\boldsymbol c})$. Then $\mathbf{u} \in \mathsf{A}_{\beta_{\boldsymbol c}}(\Theta_{\boldsymbol c})^N$.*

**Corollary 2.3** *Let $\mathbf{u} \in \mathsf{K}^2_{\boldsymbol\beta}(\Omega)^N$ satisfy $L\mathbf{u} \in \mathsf{A}_{\boldsymbol\beta+2}(\Omega)^N$ and $B_k\mathbf{u} \in \mathsf{A}_{\boldsymbol\beta+m_k+1/2}(\partial\Omega)$. Then $\mathbf{u} \in \mathsf{A}_{\boldsymbol\beta}(\Omega)^N$.*

We could apply this result to the Maxwell solution $\mathbf{u}$ of problem (1.1) if $\mathbf{f}$ belonged to an analytic space, say $\mathsf{A}_0(\Omega)^2$. Indeed, from the equivalent formulation (1.2), we can see that $\mathbf{u}$ is solution of the elliptic boundary value problem, with $L$ the diagonal Laplace operator:

$$L\mathbf{u} = -\mathbf{f} \quad \text{in} \;\; \Omega, \qquad \mathbf{u} \times \mathbf{n} = 0, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{on} \;\; \partial\Omega. \tag{2.5}$$

Using the results of [20], we find that the strongest singularity of $\mathbf{u}$ at the corner $\boldsymbol c$ has the exponent $\nu = \pi/\omega_{\boldsymbol c} - 1$, and, hence, that $\mathbf{u}$ belongs to the weighted space $K^2_{-\boldsymbol\beta}(\Omega)$ for any $\boldsymbol\beta = (\beta_{\boldsymbol c})$ with $0 \leq \beta_{\boldsymbol c} < \min\{2, \pi/\omega_{\boldsymbol c}\}$. As each component of $\mathbf{f}$ belongs to $\mathsf{A}_0(\Omega)$ which is contained in $\mathsf{A}_{-\boldsymbol\beta+2}(\Omega)$, the shift theorem yields that $\mathbf{u}$ belongs to $\mathsf{A}_{-\boldsymbol\beta}(\Omega)^2$.

But the space $\mathsf{A}_{-\boldsymbol\beta}(\Omega)^2$ *is not a subspace of the variational space* $X^{\boldsymbol\gamma}_N(\Omega)$ for any relevant choice of $\boldsymbol\gamma$, because the curls of its elements do not belong to $\mathsf{L}^2(\Omega)$ in general. That is why we have to take advantage of a splitting of $\mathbf{u}$ in the form of a singular gradient part and a "regular" part.

## 2.d The Dirichlet problem for the Laplace operator

Consider a right hand side $f \in \mathsf{K}^0_{-\delta}(\Omega)$ for $\delta \in [0,1)$ (for $\delta = 0$ in particular $f \in \mathsf{L}^2(\Omega)$) and the solution $u$ of the Dirichlet problem

$$-\Delta u = f \quad \text{in} \ \ \Omega, \qquad u \in \mathsf{H}^1_0(\Omega). \tag{2.6}$$

Denote by $S_{\boldsymbol{c},k}$ the singularities $r_{\boldsymbol{c}}^{k\pi/\omega_{\boldsymbol{c}}} \sin(k\pi\theta_{\boldsymbol{c}}/\omega_{\boldsymbol{c}})$, $k \in \mathbb{N}$, of problem (2.6) at the corner $\boldsymbol{c}$. From KONDRAT'EV [31], we obtain a decomposition of $u$ at each corner $\boldsymbol{c} \in \mathscr{C}$

$$\Big(u - \sum_{\substack{k\geq 1\\ k\pi/\omega_{\boldsymbol{c}}<1+\delta}} d_{\boldsymbol{c},k} S_{\boldsymbol{c},k}\Big)\Big|_{\Theta_{\boldsymbol{c}}} \in \mathsf{K}^2_{-2-\delta+\varepsilon}(\Theta_{\boldsymbol{c}}) \quad \forall \varepsilon > 0 \tag{2.7}$$

(here $\varepsilon$ can be omitted if no exponent $k\pi/\omega_{\boldsymbol{c}}$ equals $1+\delta$). The coefficients $d_{\boldsymbol{c},k}$ depend continuously on $f \in \mathsf{K}^0_{-\delta}(\Omega)$. In particular, for $\delta = 0$, $u\big|_{\Theta_{\boldsymbol{c}}}$ belongs to $\mathsf{K}^2_{-2}(\Theta_{\boldsymbol{c}}) \subset \mathsf{H}^2(\Theta_{\boldsymbol{c}})$ if $\boldsymbol{c}$ is a convex corner, and $(u - d_{\boldsymbol{c},1}S_{\boldsymbol{c},1})\big|_{\Theta_{\boldsymbol{c}}}$ belongs to $\mathsf{K}^2_{-2}(\Theta_{\boldsymbol{c}})$ if $\boldsymbol{c}$ is a non-convex corner, i.e. $\boldsymbol{c} \in \mathscr{C}_\star$. Let $\delta_\Delta$ be defined as

$$\delta_\Delta = \min\Big\{1\ ,\ \min_{\boldsymbol{c}\in\mathscr{C}\setminus\mathscr{C}_\star} \big(\frac{\pi}{\omega_{\boldsymbol{c}}} - 1\big)\ ,\ \min_{\boldsymbol{c}\in\mathscr{C}_\star} \big(\frac{2\pi}{\omega_{\boldsymbol{c}}} - 1\big)\Big\}. \tag{2.8}$$

For any $\delta \in [0,\delta_\Delta)$, if $f \in \mathsf{K}^0_{-\delta}(\Omega)$, $u\big|_{\Theta_{\boldsymbol{c}}}$ belongs to $\mathsf{K}^2_{-2-\delta}(\Theta_{\boldsymbol{c}})$ if $\boldsymbol{c}$ is convex, and $(u - d_{\boldsymbol{c},1}S_{\boldsymbol{c},1})\big|_{\Theta_{\boldsymbol{c}}}$ belongs to $\mathsf{K}^2_{-2-\delta}(\Theta_{\boldsymbol{c}})$ if not.

We obtain a global decomposition of $u$ on the whole domain $\Omega$ by an extension of the singular functions: Let $\chi_{\boldsymbol{c}}$ be a smooth function which is $\equiv 1$ in $\Theta_{\boldsymbol{c}}$ and $\equiv 0$ outside $\Theta'_{\boldsymbol{c}}$. Let us define $\widetilde{S}_{\boldsymbol{c}}$ by extending $\chi_{\boldsymbol{c}} S_{\boldsymbol{c},1}$ by zero outside $\Theta'_{\boldsymbol{c}}$. Then

$$\forall \delta \in [0,\delta_\Delta) \quad \text{and} \quad f \in \mathsf{K}^0_{-\delta}(\Omega), \qquad u - \sum_{\boldsymbol{c}\in\mathscr{C}_\star} d_{\boldsymbol{c},1}\widetilde{S}_{\boldsymbol{c}} \in \mathsf{K}^2_{-2-\delta}(\Omega). \tag{2.9}$$

We easily check that $\widetilde{S}_{\boldsymbol{c}}$ belongs to $\mathsf{K}^2_{-1-\boldsymbol{\beta}}(\Omega)$ for any $\boldsymbol{\beta}$ with $0 \leq \beta_{\boldsymbol{c}} < \min\{2, \pi/\omega_{\boldsymbol{c}}\}$. But of course, $\widetilde{S}_{\boldsymbol{c}}$ is not analytic inside $\Omega$. That is why we need a proof for

**Lemma 2.4** *For any $\boldsymbol{c} \in \mathscr{C}_\star$, there exists a function $S_{\boldsymbol{c}} \in \mathsf{H}^1_0(\Omega)$ which also belongs to $\mathsf{A}_{-1-\boldsymbol{\beta}}(\Omega)$ for any $\boldsymbol{\beta} = (\beta_{\boldsymbol{c}})$ with $0 \leq \beta_{\boldsymbol{c}} < \min\{2, \pi/\omega_{\boldsymbol{c}}\}$, such that for any $\delta \in [0,\delta_\Delta)$:*

(i) $\Delta S_{\boldsymbol{c}} \in \mathsf{A}_{-\delta}(\Omega)$,

(ii) $(S_{\boldsymbol{c}} - S_{\boldsymbol{c},1})\big|_{\Theta_{\boldsymbol{c}}} \in \mathsf{K}^2_{-2-\delta}(\Theta_{\boldsymbol{c}})$,

(iii) *For any corner $\boldsymbol{c}' \neq \boldsymbol{c}$, $S_{\boldsymbol{c}}\big|_{\Theta_{\boldsymbol{c}'}} \in \mathsf{K}^2_{-2-\delta}(\Theta_{\boldsymbol{c}'})$.*

PROOF. The function $\widetilde{S}_{\boldsymbol{c}}$ belongs to $\mathsf{K}^2_{-1-\boldsymbol{\beta}}(\Omega)$, satisfies *(ii)-(iii)* and the relaxed version of *(i)* : $\Delta\widetilde{S}_{\boldsymbol{c}} \in \mathsf{L}^2(\Omega)$.

Let us embed $\Omega$ in a square $Q$, extend $\Delta\widetilde{S}_{\boldsymbol{c}}$ by zero and denote this extension by $f_{\boldsymbol{c}}$. Let $\psi^{\boldsymbol{c};n}$ be the $L^2(Q)$ projection of $f_{\boldsymbol{c}}$ on the space $\mathbb{Q}^n(Q)$ of polynomials of partial degree $\leq n$ and let $\varphi^{\boldsymbol{c};n} \in \mathsf{H}^1_0(\Omega)$ be the solution of the Dirichlet problem $\Delta\varphi^{\boldsymbol{c};n} = \psi^{\boldsymbol{c};n}\big|_\Omega$.

As $\psi^{\boldsymbol{c}\,;\,n}\big|_\Omega \to \Delta\widetilde{S}_{\boldsymbol{c}}$ in $\mathsf{L}^2(\Omega)$ as $n\to\infty$, then $\varphi^{\boldsymbol{c}\,;\,n} \to \widetilde{S}_{\boldsymbol{c}}$ in the domain of $\Delta$. Therefore the coefficients $d^{\boldsymbol{c}\,;\,n}_{\boldsymbol{c}',1}$ such that

$$\forall \boldsymbol{c}'\in\mathscr{C}_\star,\quad (\varphi^{\boldsymbol{c}\,;\,n} - d^{\boldsymbol{c}\,;\,n}_{\boldsymbol{c}',1} S_{\boldsymbol{c},1})\big|_{\Theta_{\boldsymbol{c}'}} \in \mathsf{H}^2(\Theta_{\boldsymbol{c}'})$$

satisfy as $n\to\infty$

$$d^{\boldsymbol{c}\,;\,n}_{\boldsymbol{c}',1} \longrightarrow 1 \quad\text{if}\quad \boldsymbol{c}'=\boldsymbol{c} \qquad\text{and}\qquad d^{\boldsymbol{c}\,;\,n}_{\boldsymbol{c}',1} \longrightarrow 0 \quad\text{if}\quad \boldsymbol{c}'\neq\boldsymbol{c}$$

Therefore for $n$ large enough, the matrix $\big(d^{\boldsymbol{c}\,;\,n}_{\boldsymbol{c}',1}\big)_{\boldsymbol{c}\in\mathscr{C}_\star\,;\,\boldsymbol{c}'\in\mathscr{C}_\star}$ is non-singular. For such an $n$, there exists for each $\boldsymbol{c}\in\mathscr{C}_\star$ a linear combination

$$S_{\boldsymbol{c}} = \sum_{\boldsymbol{c}'\in\mathscr{C}_\star} \lambda^{\boldsymbol{c}'\,;\,n}\varphi^{\boldsymbol{c}'\,;\,n} \qquad\text{such that}\qquad \forall\boldsymbol{c}''\in\mathscr{C}_\star,\quad \sum_{\boldsymbol{c}'\in\mathscr{C}_\star}\lambda^{\boldsymbol{c}'\,;\,n} d^{\boldsymbol{c}'\,;\,n}_{\boldsymbol{c}'',1} = \delta_{\boldsymbol{c}\,;\,\boldsymbol{c}''}.$$

Since $\Delta S_{\boldsymbol{c}}$ is a polynomial on $\Omega$, it belongs to $\mathsf{A}_{-\delta}(\Omega)$ for any $\delta<1$. We easily check the other properties ***(ii)-(iii)***. Finally, we can see that $S_{\boldsymbol{c}}$ belongs to $K^2_{-1-\boldsymbol{\beta}}(\Omega)$ for any $\boldsymbol{\beta}=(\beta_{\boldsymbol{c}})$ with $0\le\beta_{\boldsymbol{c}}<\min\{2,\pi/\omega_{\boldsymbol{c}}\}$ . Since $\Delta S_{\boldsymbol{c}}$ belongs to $\mathsf{A}_{-\delta}(\Omega)$ which is contained in $\mathsf{A}_{1-\boldsymbol{\beta}}(\Omega)$, the shift theorem gives the analytic regularity $\mathsf{A}_{-1-\boldsymbol{\beta}}(\Omega)$ for $S_{\boldsymbol{c}}$. ∎

As a corollary of Lemma 2.6 and of the shift theorem we obtain

**Proposition 2.5** *For all $\delta\in[0,\delta_\Delta)$ and for all $f\in\mathsf{A}_{-\delta}(\Omega)$ there holds:*

$$u - \sum_{\boldsymbol{c}\in\mathscr{C}_\star} d_{\boldsymbol{c},1} S_{\boldsymbol{c}} \in \mathsf{A}_{-2-\delta}(\Omega).$$

### 2.e Principal singularities of Maxwell solutions

Let $\mathbf{f}\in\mathbf{L}^2(\Omega)$ with $\operatorname{div}\mathbf{f}=0$, and let $\mathbf{u}$ be the solution of problem (1.1) (or, equivalently, of problem (1.2)). In [20] the singularities at the corners of $\Omega$ are described thoroughly: For $\boldsymbol{c}\in\mathscr{C}$ the associated Maxwell singular functions are the gradients of the Laplace singularities $\mathbf{grad}\, S_{\boldsymbol{c},k}$ and other fields $\mathbf{T}_{\boldsymbol{c},k}$ of the form $r_{\boldsymbol{c}}^{k\pi/\omega_{\boldsymbol{c}}}\boldsymbol{\psi}'(\theta_{\boldsymbol{c}})$, — note that $\mathbf{grad}\, S_{\boldsymbol{c},k}$ has the form $r_{\boldsymbol{c}}^{k\pi/\omega_{\boldsymbol{c}}-1}\boldsymbol{\psi}(\theta_{\boldsymbol{c}})$.

For $\mathbf{u}$ as above, there exist coefficients $d_{\boldsymbol{c},k}$ and $d'_{\boldsymbol{c},k}$ such that (here and below, we use boldface letters for spaces of vector functions)

$$\Big(\mathbf{u} - \sum_{\substack{k\ge1\\ k\pi/\omega_{\boldsymbol{c}}<2}} d_{\boldsymbol{c},k}\,\mathbf{grad}\, S_{\boldsymbol{c},k} - \sum_{\substack{k\ge1\\ k\pi/\omega_{\boldsymbol{c}}<1}} d'_{\boldsymbol{c},k}\mathbf{T}_{\boldsymbol{c},k}\Big)\Big|_{\Theta_{\boldsymbol{c}}} \in \mathbf{K}^2_{-2+\varepsilon}(\Theta_{\boldsymbol{c}}) \quad \forall\varepsilon>0. \tag{2.10}$$

The singularities $\mathbf{grad}\, S_{\boldsymbol{c},k}$ belong to $\mathbf{K}^2_{-\beta_{\boldsymbol{c}}}(\Theta_{\boldsymbol{c}})$ for $\beta_{\boldsymbol{c}}<k\pi/\omega_{\boldsymbol{c}}$. The singularities $\mathbf{T}_{\boldsymbol{c},k}$ belong to $\mathbf{K}^2_{-1-\beta_{\boldsymbol{c}}}(\Theta_{\boldsymbol{c}})$ for $\beta_{\boldsymbol{c}}<k\pi/\omega_{\boldsymbol{c}}$. Thus we check that all $\mathbf{grad}\, S_{\boldsymbol{c},k}$ for $k\ge2$ or for $k=1$ when $\boldsymbol{c}\notin\mathscr{C}_\star$ and all $\mathbf{T}_{\boldsymbol{c},k}$ belong to $\mathbf{K}^2_{-1-\delta}(\Omega)$ for any $\delta$, $0<\delta<\delta_\Delta$. Thus we deduce from (2.10) that

$$\forall\delta\in(0,\delta_\Delta),\quad \Big(\mathbf{u} - d_{\boldsymbol{c},1}\,\mathbf{grad}\, S_{\boldsymbol{c},1}\Big)\Big|_{\Theta_{\boldsymbol{c}}} \in \mathbf{K}^2_{-1-\delta}(\Theta_{\boldsymbol{c}}) \quad\text{if}\quad \boldsymbol{c}\in\mathscr{C}_\star \tag{2.11}$$

and $\mathbf{u}\big|_{\Theta_{\boldsymbol{c}}} \in \mathbf{K}^2_{-1-\delta}(\Theta_{\boldsymbol{c}})$ otherwise.

Setting $\varphi := \sum_{\boldsymbol{c}\in\mathscr{C}_\star} d_{\boldsymbol{c},1}\, S_{\boldsymbol{c}}$ (with $S_{\boldsymbol{c}}$ the functions defined in Lemma 2.4) we have obtained a global version of (2.11) on the whole domain $\Omega$:

**Lemma 2.6** *Let* $\mathbf{f} \in \mathbf{L}^2(\Omega)$ *with* $\operatorname{div}\mathbf{f} = 0$, *and let* $\mathbf{u}$ *be the solution of problem* (1.1). *There exists* $\varphi \in \mathsf{H}^1_0(\Omega)$ *which also belongs to* $\mathsf{A}_{-1-\boldsymbol{\beta}}(\Omega)$ *for any* $\boldsymbol{\beta} = (\beta_{\boldsymbol{c}})$ *with* $0 \le \beta_{\boldsymbol{c}} < \min\{2, \pi/\omega_{\boldsymbol{c}}\}$ *such that*

$$\forall\delta \in (0, \delta_\Delta), \quad \mathbf{u} - \mathbf{grad}\,\varphi \in \mathbf{K}^2_{-1-\delta}(\Omega).$$

## 2.f Analytic regularity of Maxwell solutions

The main result of this section is the regularity of $\mathbf{u}$ when $\mathbf{f}$ belongs to the analytic weighted space $\mathbf{A}_0(\Omega)$.

**Theorem 2.7** *Let* $\mathbf{f} \in \mathbf{A}_0(\Omega)$ *with* $\operatorname{div}\mathbf{f} = 0$. *Then the solution* $\mathbf{u}$ *of problem* (1.1) *splits as*

$$\mathbf{u} = \mathbf{grad}\,\varphi + \mathbf{w} \qquad \textit{with} \quad \varphi \in \mathsf{H}^1_0 \cap \mathsf{A}_{-1-\boldsymbol{\beta}}(\Omega) \quad \textit{and} \quad \mathbf{w} \in \mathbf{A}_{-1-\delta}(\Omega) \tag{2.12}$$

*for any* $\boldsymbol{\beta} = (\beta_{\boldsymbol{c}})$ *with* $0 \le \beta_{\boldsymbol{c}} < \min\{2, \pi/\omega_{\boldsymbol{c}}\}$ *and for any* $\delta \in (0, \delta_\Delta)$.

PROOF. The existence and regularity of $\varphi$ is known from Lemma 2.6. Let $L$ be the diagonal Laplace operator. Recall that $\mathbf{u}$ solves problem (2.5). Thanks to property *(i)* in Lemma 2.4, $\Delta\varphi \in \mathsf{A}_{-\delta}(\Omega)$. We obtain that $L\mathbf{w} = \mathbf{f} - \mathbf{grad}\,\Delta\varphi$ belongs to $\mathbf{A}_{1-\delta}(\Omega)$. Moreover, $\mathbf{w}$ satisfies the same essential boundary conditions as $\mathbf{u}$, i.e. $\mathbf{w}\times\mathbf{n} = 0$ on $\partial\Omega$ and, since $\operatorname{div}\mathbf{w} = -\Delta\varphi$, $\operatorname{div}\mathbf{w}\big|_{\partial\Omega}$ belongs to $\mathsf{A}_{\frac12-\delta}(\partial\Omega)$. Since $\mathbf{w}$ already belongs to $\mathbf{K}^2_{-1-\delta}(\Omega)$, the shift theorem yields that $\mathbf{w} \in \mathbf{A}_{-1-\delta}(\Omega)$. ∎

Thus the main singularities are written as a gradient, i.e. their curl is zero. This idea can already be found in [11], but its application in the framework of weighted analytic spaces is new.

Let us fix a weight $\boldsymbol{\gamma}$ convenient for the weighted regularization: $1 - \pi/\omega_{\boldsymbol{c}} < \gamma_{\boldsymbol{c}}$ and $\gamma_{\boldsymbol{c}} \in [0,1]$, *cf* (1.4). The main property of such an exponent is that $\boldsymbol{\beta} := 1 - \boldsymbol{\gamma}$ satisfies the conditions of Theorem 2.7, therefore $\varphi$ belongs to $\mathsf{A}_{\boldsymbol{\gamma}-2}(\Omega)$. Thus, for a Maxwell solution $\mathbf{u}$ satisfying the splitting (2.12), there holds

$$\begin{aligned}
\|\mathbf{u}\|_{\mathsf{X}^{\boldsymbol{\gamma}}_N(\Omega)} &\le \|\Delta\varphi\|_{\mathsf{K}^0_{\boldsymbol{\gamma}}(\Omega)} + \|\varphi\|_{\mathsf{H}^1(\Omega)} + \|\mathbf{w}\|_{\mathsf{X}^{\boldsymbol{\gamma}}_N(\Omega)} \\
&\le \|\varphi\|_{\mathsf{K}^2_{\boldsymbol{\gamma}-2}(\Omega)} + \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \\
&\le \|\varphi\|_{\mathsf{K}^2_{\boldsymbol{\gamma}-2}(\Omega)} + \|\mathbf{w}\|_{\mathbf{K}^1_{-1}(\Omega)}\,.
\end{aligned}$$

We have obtained

**Corollary 2.8** *Let* $\boldsymbol{\gamma}$ *be a weight satisfying* (1.4). *Let* $\delta_{\boldsymbol{\gamma}}$ *be the positive number,* cf (2.8)*:*

$$\delta_{\boldsymbol{\gamma}} = \min\Big\{\delta_\Delta\,,\ \min_{\boldsymbol{c}\in\mathscr{C}}\big(\frac{\pi}{\omega_{\boldsymbol{c}}} + \gamma_{\boldsymbol{c}} - 1\big)\Big\}. \tag{2.13}$$

*Let* $\mathsf{f} \in \mathsf{A}_0(\Omega)^2$ *with* $\operatorname{div} \mathsf{f} = 0$. *Then the solution* $\mathsf{u}$ *of problem* (1.1) *splits as*

$$\mathsf{u} = \mathbf{grad}\,\varphi + \mathsf{w} \qquad \textit{with} \quad \varphi \in \mathsf{H}^1_0 \cap \mathsf{A}_{\boldsymbol{\gamma}-2-\delta}(\Omega) \quad \textit{and} \quad \mathsf{w} \in \mathbf{A}_{-1-\delta}(\Omega) \tag{2.14}$$

*for any* $\delta \in (0, \delta_{\boldsymbol{\gamma}})$. *Moreover we have the estimate of the energy norm of* $\mathsf{u}$

$$\|\mathsf{u}\|_{\mathsf{X}^{\gamma}_N(\Omega)} \leq \|\varphi\|_{\mathsf{K}^2_{\boldsymbol{\gamma}-2}(\Omega)} + \|\mathsf{w}\|_{\mathbf{K}^1_{-1}(\Omega)}\,. \tag{2.15}$$

**Remark.** If we take $\boldsymbol{\gamma} = 1$, we have $\delta_{\boldsymbol{\gamma}} = \delta_{\Delta}$.

# 3 Geometric meshes

We address in this section the principles which have to be satisfied by the geometric meshes on which $hp$-FEM spaces are constructed. Our definitions are so general as to cover geometric meshes arising in practice and, in particular, all earlier definitions given e.g. in [29, 9, 10, 7] and the references there. Geometric meshes based on our principles are realized in [25]. Not all meshes satisfying the axioms of this section will be suitable for our $hp$ approximation schemes, however: implicit conditions on the mesh stemming from the axioms on function spaces and interpolation operators of Section 4 below may have to be imposed. In some cases, the algebraic and analytic conditions on the $hp$-FE spaces can lead to conditions of purely geometric nature on the mesh. An example for this is Lemma 6.2 below for bilinearly mapped, quadrilateral elements.

We illustrate our definitions by three examples of geometric meshes on a L-shaped domain, in Figures 1, 2 and 3 corresponding to three categories **(a)**, **(b)**, and **(c)** of $hp$-FE spaces, respectively, for which we eventually give complete proofs.

## 3.a Meshes and layers

From Theorem 2.7, we know that for $\mathsf{f} \in \mathbf{A}_0(\Omega)$ the solution $\mathsf{u}$ is analytic in $\overline{\Omega}\backslash\mathscr{C}$. More precisely, for each $x \in \overline{\Omega}\backslash\mathscr{C}$, the analytic Birman-Solomyak decomposition (2.12) yields that the convergence radius of the Taylor series of $\mathsf{w}$ and $q$ at $x$ can be bounded from below by a constant times the distance from $x$ to $\mathscr{C}$. As a consequence, in any domain $K$ such that $\overline{K} \subset \Omega\backslash\mathscr{C}$, the functions $\mathsf{w}$ and $q$ can be approximated by polynomials of degree $p$ in $K$ with rate $\exp(-bp)$ where $b > 0$ depends on the *ratio of* $\inf_{x\in K} r_{\mathbf{c}}(x)$ *versus* $\operatorname{diam}(K)$. The principle underlying $hp$-FEM is to keep this ratio uniformly bounded from above and below.

For this, we consider mesh families $\mathfrak{M} = \big(\mathfrak{M}^p\big)_{p\in\mathbb{N}}$ indexed by the integer $p$ which corresponds to the degree of the reference polynomial spaces, and such that, as $p$ increases to $p+1$, only the "layer" of elements close to the corners is subdivided.

We adopt the following conventions.

A <u>mesh</u> $\mathfrak{M}$ on $\Omega$ is a finite set of (open) disjoint elements $K$ such that $\bigcup_{K\in\mathfrak{M}} \overline{K} = \overline{\Omega}$. *Note that, at this stage, we do not require the "usual" conformity conditions on the intersection of the elements* $\overline{K}$*, considering "hanging nodes" as admissible, see below, Section 6.a.*

An <u>element</u> $K \in \mathfrak{M}$ is either a convex quadrilateral with straight sides or a triangle, hence

$$K = F_K(I^2), \text{ with } F_K \in \big(\mathbb{Q}^1\big)^2, \text{ or } \ K = F_K(S^2), \text{ with } F_K \in \big(\mathbb{P}^1\big)^2$$

with $F_K$ a diffeomorphism, and with the reference elements $\widehat{K} = I^2$ (unit square) or $\widehat{K} = S^2$ (unit simplex).

Each family $\mathfrak{M}$ consists of an infinite sequence of disjoint layers $\mathfrak{L}^p$, $p \geq 0$ and an infinite sequence of nested terminal layers $\mathfrak{T}^p$, $p \geq 1$ such that for each $p \geq 1$

$$(\mathrm{M}_0) \qquad \begin{cases} \mathfrak{M}^p := \mathfrak{L}^0 \cup \mathfrak{L}^1 \cup \ldots \cup \mathfrak{L}^{p-1} \cup \mathfrak{T}^p \quad \text{(disjoint union)} \quad \text{is a mesh on } \Omega \\ \forall j \geq 0, \quad \forall K \in \mathfrak{L}^j, \quad \forall K' \in \mathfrak{L}^{j+2}, \quad \overline{K} \cap \overline{K'} = \emptyset, \\ \forall \boldsymbol{c} \in \mathscr{C}, \quad \exists K \in \mathfrak{T}^p, \quad \boldsymbol{c} \in \overline{K} \end{cases}$$

The hypothesis of separation between the layers $\mathfrak{L}^j$ and $\mathfrak{L}^{j+2}$ is not restrictive. It is introduced mainly for later convenience.

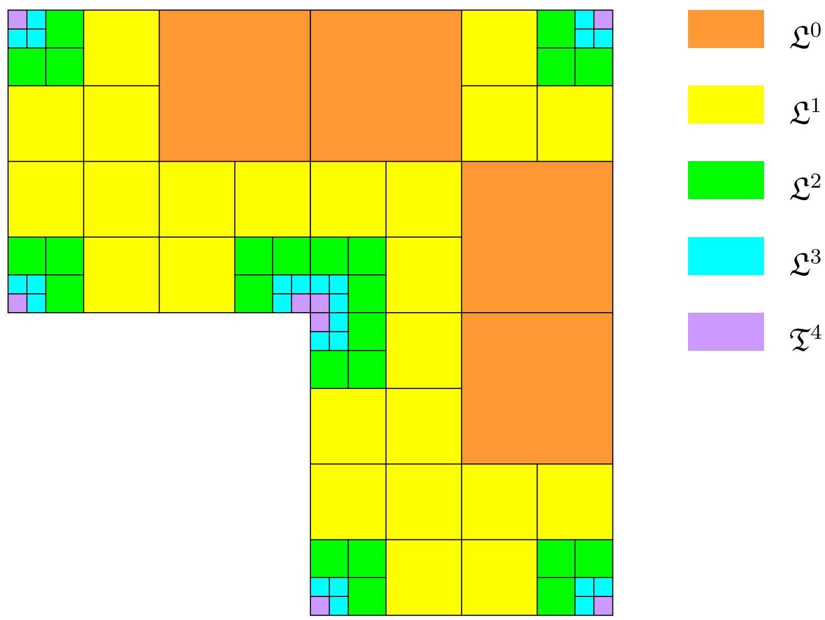


Figure 1: Mesh of squares with hanging nodes – case **(a)**.

### 3.b $\sigma$-geometric meshes

Now we quantify the properties of the elements $K$ relating to their position with respect to the corners. For doing this, we fix a covering $(\Theta_0, \Theta_{\boldsymbol{c}})$ of $\Omega$ as in (2.1). We denote by $x_K$ the center of $K$ and by $i_K$ the following localization index:

If $x_K$ belongs to $\Theta_0$, then $i_K := 0$

If $x_K \notin \Theta_0$, $\exists \boldsymbol{c} \in \mathscr{C}$ unique, $x_K \in \Theta_{\boldsymbol{c}}$; then $i_K := \boldsymbol{c}$.

Let $d_K$ be the following distance parameter:

If $i_K = 0$, then $d_K := 1$

If $i_K = \boldsymbol{c}$, then $d_K := r_{\boldsymbol{c}}(x_K)$.

Finally we denote by $H_K$ the homothety with center $x_K$ and ratio $d_K$, that is

$$H_K(x) = x_K + d_K(x - x_K)$$

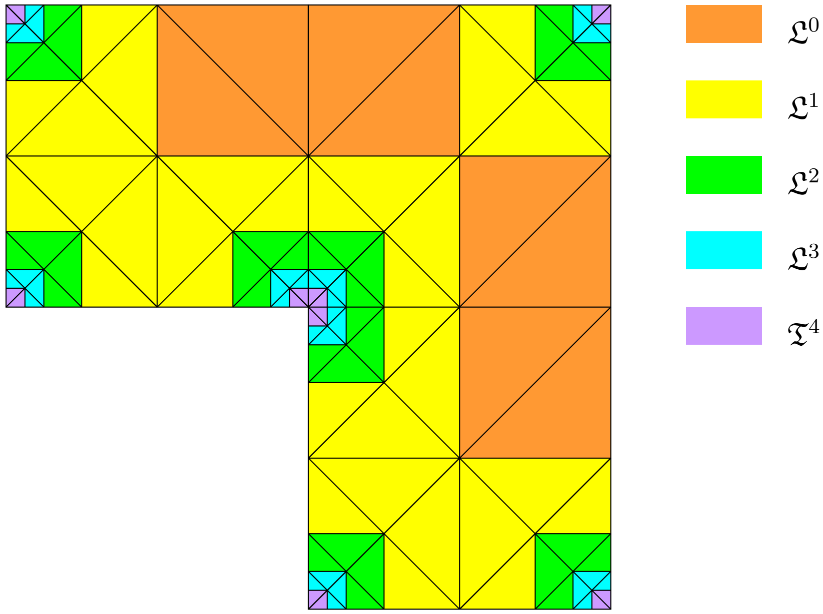


Figure 2: Mesh with triangle elements – case **(b)**.

and by $\breve{K}$ the "semi-reference" element $\breve{K} = H_K^{-1}(K)$ with the associated chart

$$\breve{F}_K := H_K^{-1} \circ F_K \qquad (\text{i.e.} \quad F_K = H_K \circ \breve{F}_K)$$

Let $\breve{a}_{K,i}$ be the coefficients of $\breve{F}_K$ and $\breve{J}_K$ be its Jacobian determinant $\breve{J}_K = \det D\breve{F}_K$.

**Definition 3.1** Let $\mathfrak{M} = \big(\mathfrak{M}^p\big)_{p\in\mathbb{N}}$ be a family of meshes with the structure $(\mathrm{M}_0)$. Let $\sigma \in (0,1)$. The family $\mathfrak{M}$ is *geometric* with grading factor $\sigma$ (*"$\sigma$-family"* for short), if there exists a regularity constant $\kappa > 1$ such that the following conditions $(\mathrm{M}_1)$–$(\mathrm{M}_3)$ are satisfied:

$(\mathrm{M}_1)$ The family of scaled diffeomorphisms $\big(\breve{F}_K\big)_{K\in\cup_p\mathfrak{M}^p}$ is a uniform $\kappa$-family of mappings:

$$|\breve{J}_K| \geq \kappa^{-1} \text{ on } \widehat{K} \qquad \text{and} \qquad \breve{a}_{K,i} \leq \kappa.$$

$(\mathrm{M}_2)$ $\forall p \geq 1$, $\forall K \in \mathfrak{T}^p$, $\quad \kappa^{-1}\sigma^p \leq d_K \leq \kappa\sigma^p$.

$(\mathrm{M}_3)$ There exists a larger covering $(\Theta'_0, \Theta'_{\boldsymbol{c}})$, *cf* §2.a, of $\Omega$ such that $\forall K \in \cup_p\mathfrak{M}^p$
If $i_K = 0$, then $K \subset \Theta'_0$,
If $i_K = \boldsymbol{c}$, then $K \subset \Theta'_{\boldsymbol{c}}$ and

$$\kappa^{-1}d_K \leq r_{\boldsymbol{c}}\big|_K \leq \kappa d_K \quad \text{if } K \in \cup_j\mathfrak{L}^j, \qquad r_{\boldsymbol{c}}\big|_K \leq \kappa d_K \quad \text{if } K \in \cup_j\mathfrak{T}^j.$$

∎

We have written the conditions on the mesh families in the easiest way for their application to error estimates. It is also interesting to draw consequences of these conditions on the layers $\mathfrak{L}^j$ to figure out the structure of the meshes.

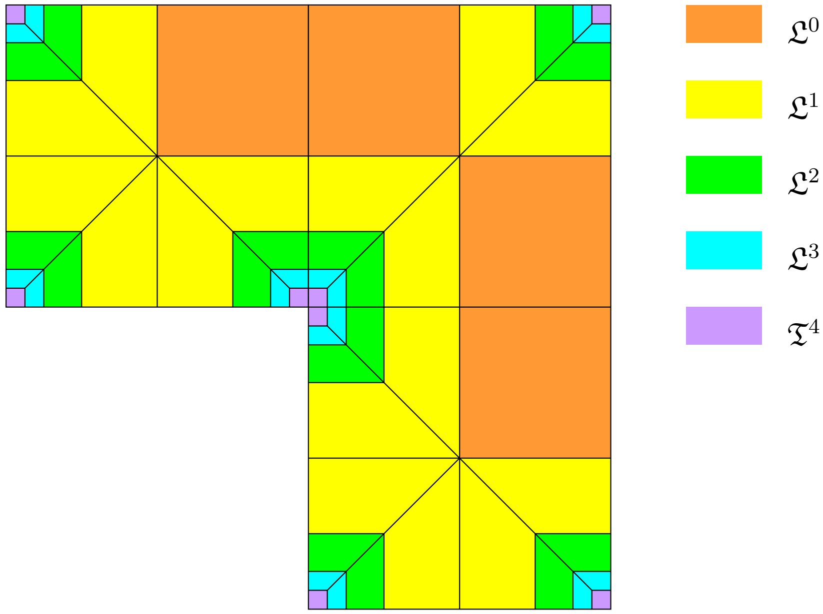


Figure 3: Mesh with $Q^1$ elements (trapezia) – case **(c)**.

Let $\mathfrak{M} = \left(\mathfrak{M}^p\right)_{p\in\mathbb{N}}$ be a $\sigma$-family of meshes. Then there exists $j_0$ such that for all $j \geq j_0$, and for all $K \in \mathfrak{L}^j$ the intersection $K \cap \Theta_0'$ is empty. Thus, for all $j \geq j_0$ and for all $K \in \mathfrak{L}^j$, the localization index of $K$ is a corner $\boldsymbol{c}$ and there holds

$$\kappa^{-2}\sigma^j \leq r_{\boldsymbol{c}}\big|_K \leq \kappa^2\sigma^j. \tag{3.1}$$

### 3.c Patches

The construction of $hp$-interpolants in Section 4 will be a two-step procedure. In the first step, one constructs a basic interpolant defined elementwise which will in general not satisfy inter-element continuity conditions. The construction may even start from a projection operator that has no pointwise interpolation properties at all and which is then corrected on the element level in order to interpolate a certain number of derivatives at the nodes.

In the second step, conformity, i.e. inter-element continuity, is achieved by the construction of interface correctors that are defined on patches of elements (2 or 3 elements, in general) that share an interface.

We now define the hypotheses that the geometry of such patches will have to satisfy.

**Definition 3.2**
*(i)* We call patch a subset $P$ of a mesh $\mathfrak{M}$ such that the interior $U_P$ of $\cup_{K\in P}\overline{K}$ is connected.

*(ii)* Let $A$ be a subset of edges of elements $K \in \mathfrak{M}$. We say that the patch $P$ is associated with $A$ if $\cup_{a\in A}\, a$ is contained in $U_P$.

*(iii)* For each patch $P$, we choose a center point $x_P \in U_P$, and define its localization index $i_P$ and the distance parameter $d_P$ as before for $K$. We also denote by $H_P$ the homothety of center $x_P$ and ratio $d_P$. ∎

For a $\sigma$-family $\left(\mathfrak{M}^p\right)_p$, we denote by $\mathfrak{A}^p$ the set of (open) edges $a$ of all elements $K \in \mathfrak{L}^0 \cup \ldots \cup \mathfrak{L}^{p-2}$ such that $a \cap \partial\Omega = \emptyset$.

**Definition 3.3** We call admissible family of patches associated with the $\sigma$-family $\left(\mathfrak{M}^p\right)_p$ a family $\mathfrak{P} = \left(\mathfrak{P}^p\right)_p$ where for any $p \geq 2$, we have the following properties:

($\mathrm{P}_1$) For each $p$, each $P \in \mathfrak{P}^p$ is a patch of $\mathfrak{M}^p \setminus \mathfrak{T}^p$, associated with a subset of edges $A = A(P) \subset \mathfrak{A}^p$ so that the $A(P)$ are mutually disjoint and $\mathfrak{A}^p$ is the union of the $A(P)$.

($\mathrm{P}_2$) There exists an integer $N$ such that for any $p \geq 2$ and each point $x \in \Omega$, $x$ belongs to at most $N$ different patches $P \in \mathfrak{P}^p$.

($\mathrm{P}_3$) There exists a larger covering $(\Theta_0', \Theta_{\boldsymbol{c}}')$, *cf* §2.a, of $\Omega$ such that $\forall P \in \cup_p \mathfrak{P}^p$
If $i_P = 0$, then $U_P \subset \Theta_0'$,
If $i_P = \boldsymbol{c}$, then $U_P \subset \Theta_{\boldsymbol{c}}'$ and

$$\kappa^{-1} d_P \leq r_{\boldsymbol{c}}\big|_{U_P} \leq \kappa d_P. \qquad \blacksquare$$

In the situation of standard conforming interfaces between elements, for any edge $a$ there exist at most two elements $K$ and $K'$ which share $a$, and the patch $P$ associated with $A = \{a\}$ is $K \cup K'$. This is the situation for our concrete cases **(b)** and **(c)**. In case **(a)**, any hanging node corresponds to a set $A$ of three edges, corresponding to a patch of three elements.

# 4 $hp$-Interpolants

In this section, we describe the general structure of the function spaces and interpolation operators that will serve to construct the finite dimensional subspaces of the energy space used as test and trial spaces, and analyze their approximation properties in weighted analytic spaces.

We want to construct conforming finite element approximations with error estimates based on the decomposition (2.14) of the solution $\mathbf{u}$ into a regular part $\mathbf{w}$ and a gradient $\mathbf{grad}\,\varphi$. This means that we need to consider globally continuous approximations for both $\mathbf{w}$ and $\mathbf{grad}\,\varphi$. We shall have to approximate $\mathbf{grad}\,\varphi$ by gradients, thus requiring $\mathscr{C}^1$ approximations for $\varphi$.

Thus we have to construct vector valued $\mathscr{C}^0$ elements and scalar $\mathscr{C}^1$ elements with the additional property that the gradients of the latter belong to the same finite dimensional space as the former. Note that in the implementation of $hp$-FEM, only the $\mathscr{C}^0$ elements will be used; the $\mathscr{C}^1$ elements are a purely theoretic tool required by our strategy of proving error estimates.

These requirements are specific to our method of approximation for Maxwell's equations and they demand a certain degree of flexibility and generality of the hypotheses for the function spaces and interpolants.

We are going to introduce our collection of axioms in the order which could be the most natural for the reader: In §4.a elementwise interpolants are defined independently on each $K$. This produces a global interpolant $\Pi^p$ on $\Omega$, but we have to modify it on the element interfaces (§4.b), at corners (§4.c), and, if essential boundary conditions have to be implemented, along the boundary of $\Omega$ (§4.e). The global operator $\Pi^p$ has to satisfy enough nodal interpolation properties to allow all the mentioned corrections *locally*. The first global interpolant has to satisfy some *analytic type interpolation estimates* (which will be proved for our examples in Section 8), and the various corrections have to satisfy *stability estimates*.

### 4.a Elementwise interpolants

For each $p \in \mathbb{N}$ and $K \in \mathfrak{M}^p$, we give ourselves a first approximation space $V_K^p$ of finite dimension and assume the existence of a linear operator

$$\Pi_K^p \;:\; \mathscr{C}^\infty(\overline{K}) \to V_K^p\,.$$

In the situation where the elementwise maps $F_K$ are affine (e.g. our concrete cases **(a)** and **(b)**) we will take $V_K^p$ as $\mathbb{Q}^p$ on parallelepipeds $K$ – the polynomials of partial degrees less than $p$ in the axes directions of $K$, or $\mathbb{P}^p$ on triangles – the polynomials of global degree less than $p$. In such a situation, it seems simpler and more usual to define the discrete spaces and the interpolants on the reference element $\widehat{K}$ and to push them forward to $K$ using the element maps $F_K : \widehat{K} \to K$:

$$F_K^* : u \mapsto F_K^* u = u \circ F_K.$$

But in more general situations than the affine mapped rectangles – our case **(c)**, we adopt the converse point of view: We start defining $\Pi_K^p$ and $V_K^p$ and transport them on the reference element $\widehat{K}$ in order to introduce an axiom providing uniform estimates: Let

$$\widehat{V}_K^p = V_K^p \circ F_K := F_K^*(V_K^p) \quad \text{and} \quad \widehat{\Pi}_K^p = F_K^* \Pi_K^p (F_K^*)^{-1}\,,$$

i.e.

$$\hat{u} = u \circ F_K \quad \text{and} \quad \widehat{\Pi}_K^p \hat{u} = (\Pi_K^p u) \circ F_K.$$

We assume the following approximation estimates in the Sobolev norm $\mathsf{H}^\ell$, where $\ell$ is fixed (and will be chosen later, see Section 6):

$(\mathrm{I}_1)$ For all $\hat{u} \in \mathscr{C}^\infty(\overline{\widehat{K}})$ and for all integer $k$ such that $\ell < k < p$

$$\|\hat{u} - \widehat{\Pi}_K^p \hat{u}\|_{\mathsf{H}^\ell(\widehat{K})} \leq \Psi_{p,k} \|\hat{u}\|_{\mathsf{H}^k(\widehat{K})} \tag{4.1}$$

where the convergence rate $\Psi_{p,k}$ does not depend on $K$.

As a consequence of $(\mathrm{I}_1)$, if moreover $(\mathrm{M}_1)$ holds, we have uniform interpolation error estimates on the *semi-reference elements* $\breve{K} = H_K^{-1}(K)$

$$\|\breve{u} - \breve{\Pi}_K^p \breve{u}\|_{\mathsf{H}^\ell(\breve{K})} \leq C\,\Psi_{p,k} \|\breve{u}\|_{\mathsf{H}^k(\breve{K})} \tag{4.2}$$

with a constant $C$ independent on $K$, $p$, $k$ and

$$\breve{u} = u \circ H_K \qquad \text{and} \qquad \breve{\Pi}_K^p \breve{u} = (\Pi_K^p u) \circ H_K.$$

**Proposition 4.1** *Let the family of meshes $\mathfrak{M} = \big(\mathfrak{M}^p\big)_{p\in\mathbb{N}}$ satisfy assumptions $(\mathrm{M}_0)$, $(\mathrm{M}_1)$ and $(\mathrm{M}_3)$. Let for any $p \geq 1$*

$$\mathfrak{O}^p := \mathfrak{L}^0 \cup \mathfrak{L}^1 \cup \ldots \cup \mathfrak{L}^{p-1} \qquad \textit{and} \qquad \Omega^p = \Omega \setminus \bigcup_{K\in\mathfrak{T}^p} \overline{K} = \mathrm{int}\big( \bigcup_{K\in\mathfrak{O}^p} \overline{K}\big).$$

*We assume moreover that assumption $(\mathrm{I}_1)$ holds. For $u \in \mathscr{C}^\infty(\overline{\Omega}^p)$ let $\Pi^p u$ be defined on each $K \in \mathfrak{O}^p$ by $\Pi^p u\big|_K = \Pi^p_K(u\big|_K)$. Then for all $\beta \in \mathbb{R}$ and all $k$ with $\ell < k < p$, we have the estimate*

$$\|u - \Pi^p u\|_{\mathsf{K}^\ell_\beta(\mathfrak{O}^p)} \leq C\, \Psi_{p,k} \|u\|_{\mathsf{K}^k_\beta(\mathfrak{O}^p)}\,. \tag{4.3}$$

*Here $\|v\|_{\mathsf{K}^\ell_\beta(\mathfrak{O}^p)}$ is the* "broken norm" $\big(\sum_{K\in\mathfrak{O}^p} \|v\big|_K\|^2_{\mathsf{K}^\ell_\beta(K)}\big)^{1/2}$*, the constant $C$ is independent of $K$, $p$, $k$, and $\ell$ and $\Psi_{p,k}$ are as in $(\mathrm{I}_1)$.*

PROOF. Let $K$ belong to $\mathfrak{O}^p$. Thanks to assumption $(\mathrm{M}_3)$, we can freeze the weight on each element. Then by the homothety $H_K$ we transport the norm to $\breve{K}$. Here we denote $u\big|_K \circ H_K$ by $\breve{u}_K$. Let us prove first the following equivalence of norms, where the equivalence constants do not depend on $K$, $m \in \mathbb{N}$, $u \in \mathscr{C}^\infty(\overline{K})$, $\beta \in \mathbb{R}$:

$$\|u\|_{\mathsf{K}^m_\beta(K)} \simeq d_K^{\beta+1} \|\breve{u}\|_{\mathsf{H}^m(\breve{K})} \tag{4.4}$$

Indeed,

$$\begin{aligned}
\|u\|^2_{\mathsf{K}^m_\beta(K)} &\simeq \sum_{|\alpha|\leq m} \|r^{\beta+|\alpha|}\partial^\alpha u\|^2_{\mathsf{L}^2(K)} \\
&\simeq \sum_{|\alpha|\leq m} d_K^{2(\beta+|\alpha|)} \|\partial^\alpha u\|^2_{\mathsf{L}^2(K)} \\
&\simeq \sum_{|\alpha|\leq m} d_K^{2(\beta+|\alpha|)}\, d_K^{2(1-|\alpha|)} \|\partial^\alpha\big(u\circ H_K\big)\|^2_{\mathsf{L}^2(\breve{K})} \\
&= \sum_{|\alpha|\leq m} d_K^{2(\beta+1)} \|\partial^\alpha \breve{u}\|^2_{\mathsf{L}^2(\breve{K})} \\
&\simeq d_K^{2(\beta+1)} \|\breve{u}\|^2_{\mathsf{H}^m(\breve{K})}\,.
\end{aligned}$$

Now we use (4.2) and (4.4) twice and obtain

$$\begin{aligned}
\|u - \Pi_p u\|_{\mathsf{K}^\ell_\beta(K)} &\simeq d_K^{\beta+1} \|\breve{u} - \breve{\Pi}^p_K \breve{u}\|_{\mathsf{H}^\ell(\breve{K})} \\
&\lesssim d_K^{\beta+1}\, \Psi_{p,k} \|\breve{u}\|_{\mathsf{H}^k(\breve{K})} \\
&\simeq \Psi_{p,k}\, \|u\|_{\mathsf{K}^k_\beta(K)}\,.
\end{aligned}$$

Squaring this inequality and summing over all $K \in \mathfrak{O}^p$, we obtain (4.3). ∎

If the bounds $\Psi_{p,k}$ have the special form (4.5), the exponential convergence rate for $hp$ methods follows immediately (Corollary 4.2). This special form for the $\Psi_{p,k}$ will be verified for our examples of $hp$-FE spaces in Section 8.

**Corollary 4.2** *Under the assumptions of Proposition* 4.1 *on the meshes and interpolation operators, we suppose moreover that the constants* $\Psi_{p,k}$ *in estimate* (4.1) *have the following bounds: There exists a constant* $c > 0$ *such that*

$$(\Psi_{p,k})^2 \leq c^k \,\frac{(p-k)!}{(p+k)!}, \quad \forall p, k > 0,\ k < p\,. \tag{4.5}$$

*Then for any* $u$ *in the analytic weighted space* $\mathsf{A}_\beta(\Omega)$ *we have the exponential convergence of the interpolation error*

$$\|u - \Pi^p u\|_{\mathsf{K}^\ell_\beta(\mathfrak{O}^p)} \leq C\, e^{-bp} \quad \textit{with} \quad b > 0 \quad \textit{independent of } p. \tag{4.6}$$

PROOF. Combining (4.3), (4.5) with (2.4) we obtain for any $k$, $\ell < k < p$

$$\|u - \Pi^p u\|^2_{\mathsf{K}^\ell_\beta(\mathfrak{O}^p)} \leq C^{2k} (k!)^2 \,\frac{(p-k)!}{(p+k)!}\,.$$

By Stirling's formula $n! \simeq n^n e^{-n}\sqrt{2\pi n}$ there exists $\delta > 0$ such that

$$C^{2k}(k!)^2 \,\frac{(p-k)!}{(p+k)!} \ \leq\ \delta^{2k}\, \frac{(p-k)^{p-k}\, k^k\, k^k}{(p+k)^{p+k}} \ =\ \Big(\frac{p-k}{p+k}\Big)^{p-k}\Big(\frac{\delta k}{p+k}\Big)^{2k}.$$

Choosing $k = p/(\delta+1)$, we obtain

$$C^{2k}(k!)^2 \,\frac{(p-k)!}{(p+k)!} = \Big(\frac{\delta k}{(\delta+2)k}\Big)^{p-k}\Big(\frac{\delta k}{(\delta+2)k}\Big)^{2k} = \Big(\frac{\delta}{\delta+2}\Big)^{p(1+\frac{1}{\delta+1})}.$$

With $b := -\log\left(\frac{\delta}{\delta+2}\right)^{(1+\frac{1}{\delta+1})/2}$ we have proved (4.6). ∎

### 4.b Interface correctors on patches

With the elementwise defined spaces $V^p_K$ and operators $\Pi^p_K$ we associate broken spaces of functions (discontinuous in general) on patches $P$ or on the whole domain $\Omega$, and the corresponding interpolation operators:

Let $\mathfrak{P} = \big(\mathfrak{P}^p\big)_{p\in\mathbb{N}}$ be an admissible family of patches. Let $p \geq 2$ and $P \in \mathfrak{P}^p$. We define

$$\mathscr{C}^\infty(P) = \prod_{K\in P} \mathscr{C}^\infty(\overline{K}) \quad \text{and} \quad V^p(P) = \prod_{K\in P} V^p_K\,.$$

These spaces are subspaces of $L^\infty(U_P)$, where we recall that $U_P$ is the interior of $\cup_{K\in P}\overline{K}$.

From our family of interpolants $\Pi^p_K : \mathscr{C}^\infty(\overline{K}) \to V^p_K$ we define the interpolation operator $\Pi^p(P) := \prod_{K\in P}\Pi^p_K$ which acts from $\mathscr{C}^\infty(P)$ into $V^p(P)$. The space $\mathscr{C}^\infty(\overline{U}_P)$ is a subspace of $\mathscr{C}^\infty(P)$. Likewise, we define the interpolation operator $\Pi^p(\mathfrak{O}^p)$ on the set $\mathfrak{O}^p$ of non terminal elements (we recall that $\mathfrak{O}^p = \mathfrak{L}^0 \cup \mathfrak{L}^1 \cup \ldots \cup \mathfrak{L}^{p-1}$).

We denote by $V^p_{\rm nod}(P)$ and $V^p_{\rm nod}(\mathfrak{O}^p)$ the image of $\mathscr{C}^\infty(\overline{U}_P)$ and $\mathscr{C}^\infty(\overline{\Omega}^p)$ by $\Pi^p(P)$ and $\Pi^p(\mathfrak{O}^p)$, respectively:

$$V^p_{\rm nod}(P) = \Pi^p(P)\big(\mathscr{C}^\infty(\overline{U}_P)\big) \quad \text{and} \quad V^p_{\rm nod}(\mathfrak{O}^p) = \Pi^p(\mathfrak{O}^p)\big(\mathscr{C}^\infty(\overline{\Omega}^p)\big). \tag{4.7}$$

In the typical case where the basic interpolants $\widehat{\Pi}^p_K$ interpolate some derivatives at the corners of the reference domain, the elements of $V^p_{\mathrm{nod}}$ will satisfy corresponding matching conditions at the nodes of the mesh. They will, in general, still be discontinuous across the edges, however. We will achieve interelement continuity by constructing interface correctors.

Let $A = A(P)$ be the set of (open) edges $a$ which is associated with the patch $P$, and let $B$ be the remaining set of edges $b$ of elements $K \in P$. Since for all $a \in A$, $a$ is contained in $U_P$, all the edges contained in $\partial U_P$ belong to $B$. For a two-element patch in particular $B$ is exactly the set of edges contained in $\partial U_P$.

Each "active" edge $a \in A$ runs between two elements of the patch $P$, hence for any $v \in V^p(P)$, the *jump* $[v]_a$ of $v$ across $a$ is well-defined.

The application of interface correctors possibly increases the degree of the local polynomial spaces. To allow for such an increase, we admit a second family of spaces $W^p_K \supset V^p_K$ together with the following axioms on local interface correctors:

**Definition 4.3** An <u>interface corrector</u> of order $d \geq 0$ for the family of interpolants $(\Pi^p_K)$ on the patch $P \in \mathfrak{P}^p$ consists of
- discrete spaces $W^p_K \supset V^p_K$ on each $K \in P$,
- an operator $R^p_P : V^p_{\mathrm{nod}}(P) \to W^p(P) := \prod_{K\in P} W^p_K$ for the correction of jumps $[v]_a$ on each edge $a \in A$, satisfying the algebraic condition $(\mathrm{I}_2)$ and the stability condition $(\mathrm{I}_3)$:

$(\mathrm{I}_2)$ For all $v \in V^p_{\mathrm{nod}}(P)$, the function $w := R^p_P v$ satisfies

$$\begin{aligned} &\forall a \in A, \quad \forall \alpha,\ |\alpha| \leq d, \qquad [\partial^\alpha w]_a &=& \ [\partial^\alpha v]_a \\ &\forall b \in B, \quad \forall \alpha,\ |\alpha| \leq d, \qquad \partial^\alpha w\big|_b &=& \ 0. \end{aligned}$$

$(\mathrm{I}_3)$ With $\breve{P} = H_P^{-1}(P)$, $\breve{v} = v \circ H_P$ and $\breve{R}^p_P = H^*_P R^p_P (H^*_P)^{-1}$, there hold the uniform estimates

$$\|\breve{R}^p_P \breve{v}\|_{\mathsf{H}^{d+1}(\breve{P})} \leq C \inf\left\{ \|\breve{v} - \breve{z}\|_{\mathsf{H}^{\ell}(\breve{P})} \ \middle|\ \ \breve{z} \in \mathsf{H}^{\ell}(U_{\breve{P}}) \right\}.$$

Here $\mathsf{H}^\ell(\breve{P})$ is the broken norm and $\ell \geq d+1$ is a fixed integer.

■

**Remark 4.4** The existence of the function $w$ in $(\mathrm{I}_2)$ implies that the jumps $[\partial^\alpha v]_a$ vanish at the common node of $a$ and $b$, for all functions $v$ in the range of $\Pi^p$. Therefore the mere existence of an interface corrector implies that the basic "interpolant" $\Pi^p_K$ has indeed some interpolation properties at the nodes, although we did not need to impose this before. In this way, hypothesis $(\mathrm{I}_2)$ is not only an explicit condition imposed on $R^p_P$, but also an implicit condition on $\Pi^p_K$. This will have to be taken into account in the construction of $\Pi^p_K$ in the examples of Section 6.

■

We obtain a similar statement to Proposition 4.1:

**Proposition 4.5** *We assume that $\mathfrak{P} = \left(\mathfrak{P}^p\right)_{p\in\mathbb{N}}$ is an admissible family of patches satisfying condition $(\mathrm{P}_3)$ and that we have a family of interpolants $(\Pi^p_K)$. If the interface corrector $R^p_P$ satisfies $(\mathrm{I}_3)$ then for any $\beta \in \mathbb{R}$ and $v \in V^p_{\mathrm{nod}}(P)$, we have the estimate*

$$\|R^p_P v\|_{\mathsf{K}^{d+1}_\beta(P)} \leq C \inf\left\{ \|v - z\|_{\mathsf{K}^{\ell}_\beta(P)} \ \middle|\ \ z \in \mathsf{H}^{\ell}(U_P) \right\}. \tag{4.8}$$

Note that the norm on the right hand side of (4.8) is a norm on the *jumps* of $v$ across the edges $a \in A$.

### 4.c Interpolants in the corner regions

We do not use the interpolants $\Pi_K^p$ when $K$ belongs to $\mathfrak{T}^p$, but the trivial approximation by zero $Z_K^p u = 0$. We need a transition from $Z_K^p$ to $\Pi_K^p$. We denote by $\widetilde{\mathfrak{T}}^{p-2} := \mathfrak{T}^p \cup \mathfrak{L}^{p-1} \cup \mathfrak{L}^{p-2}$ the extended terminal layer. We recall that $\mathfrak{O}^{p-2}$ is defined as $\mathfrak{L}^{p-3} \cup \ldots \cup \mathfrak{L}^0$ and that, in this case,

$$\mathfrak{M}^p = \widetilde{\mathfrak{T}}^{p-2} \cup \mathfrak{O}^{p-2}.$$

**Definition 4.6** The corner interpolant $Z_K^p$ is defined for all $K \in \widetilde{\mathfrak{T}}^{p-2}$ so that:

$(\mathrm{I}_4)$ For $K \in \mathfrak{T}^p \cup \mathfrak{L}^{p-1}$, $Z_K^p \equiv 0$ and for $K \in \mathfrak{L}^{p-2}$, $Z_K^p$ acts $\mathscr{C}^\infty(\overline{K}) \to V_K^p$.
For any function $v \in \mathscr{C}^\infty(\overline{\Omega}^p)$ and defined on $\Omega$, the function $w =: J^p v$ defined on all elements $K \in \mathfrak{M}^p$ by

$$\forall K \in \widetilde{\mathfrak{T}}^{p-2},\;\; w\big|_K = Z_K^p v \quad\text{and}\quad \forall K \in \mathfrak{O}^{p-2},\;\; w\big|_K = \Pi_K^p v$$

satisfies $w \in V^p_{\mathrm{nod}}(\mathfrak{O}^p)$.

$(\mathrm{I}_5)$ For some integer $m \geq \ell$ there hold the uniform stability estimates on the semi-reference element

$$\|\breve{v} - \breve{Z}_K^p \breve{v}\|_{\mathsf{H}^\ell(\breve{K})} \leq C\, \|\breve{v}\|_{\mathsf{H}^m(\breve{K})}\,.$$

where $C$ is independent of $p$ and $K$. ■

**Proposition 4.7** *Let the family of meshes $\mathfrak{M} = \big(\mathfrak{M}^p\big)$ satisfy assumptions $(\mathrm{M}_0)$, $(\mathrm{M}_1)$, $(\mathrm{M}_2)$ and $(\mathrm{M}_3)$. Assume moreover that $(\mathrm{I}_5)$ holds. For $u \in \mathscr{C}^\infty(\overline{\Omega}^p)$ defined on $\Omega$, let $Z^p u$ be defined on each $K \in \widetilde{\mathfrak{T}}^{p-2}$ by $Z^p u\big|_K = Z_K^p(u\big|_K)$.*

*Then for all $\beta, \beta' \in \mathbb{R}$ with $\beta' \leq \beta$ and all $u \in \mathsf{K}^m_{\beta'}(\Omega)$, we have the estimate*

$$\|u - Z^p u\|_{\mathsf{K}^\ell_\beta(\widetilde{\mathfrak{T}}^{p-2})} \leq C\, \sigma^{p(\beta-\beta')} \|u\|_{\mathsf{K}^m_{\beta'}(\widetilde{\mathfrak{T}}^{p-2})}\,, \tag{4.9}$$

*where the constant $C$ is independent of $p$.*

PROOF. Let $K$ belong to $\mathfrak{T}^p \cup \mathfrak{L}^{p-1}$. Here $Z_K^p u = 0$.

$$\begin{aligned}
\|u - Z^p u\|_{\mathsf{K}^\ell_\beta(K)} = \|u\|_{\mathsf{K}^\ell_\beta(K)} &\simeq \sum_{|\alpha|\leq\ell} \|r^{\beta+|\alpha|}\partial^\alpha u\|_{\mathsf{L}^2(K)} \\
&\lesssim \sup_{x\in K} r(x)^{\beta-\beta'} \sum_{|\alpha|\leq\ell} \|r^{\beta'+|\alpha|}\partial^\alpha u\|_{\mathsf{L}^2(K)} \\
&\simeq d_K^{\beta-\beta'}\, \|u\|_{\mathsf{K}^\ell_{\beta'}(K)} \\
&\simeq \sigma^{p(\beta-\beta')}\, \|u\|_{\mathsf{K}^\ell_{\beta'}(K)}\,.
\end{aligned} \tag{4.10}$$

If $K$ belongs to $\mathfrak{L}^{p-2}$ we use $(\mathrm{I}_5)$ together with the usual scaling to $\breve{K}$ and we obtain the inequality $\|u - Z^p u\|_{\mathsf{K}^\ell_\beta(K)} \le C\|u\|_{\mathsf{K}^\ell_\beta(K)}$. Then we obtain (4.9) by combining this with estimates like (4.10) above, since for $K \in \mathfrak{L}^{p-2}$ the size $d_K$ is bounded by $C\sigma^p$, *cf* $(\mathrm{M}_3)$ and (3.1). ∎

### 4.d An interpolant of class $\mathscr{C}^d$ with exponential estimates

We obtain such an interpolant $\mathcal{I}^p = \mathcal{I}^p_{(d)}$ by chaining together the previous spaces and interpolants: First recall from $(\mathrm{I}_4)$ that $J^p$ is the extension of $\Pi^p$ by $Z^p$:

$$\forall K \in \widetilde{\mathfrak{T}}^{p-2}, \quad J^p u\big|_K = Z^p_K\big(u\big|_K\big) \quad \text{and} \quad \forall K \in \mathfrak{O}^{p-2}, \quad J^p u\big|_K = \Pi^p_K\big(u\big|_K\big) \tag{4.11}$$

Recall that $\widetilde{\mathfrak{T}}^{p-2} = \mathfrak{T}^p \cup \mathfrak{L}^{p-1} \cup \mathfrak{L}^{p-2}$ and $\mathfrak{O}^{p-2} = \mathfrak{L}^{p-3} \cup \ldots \cup \mathfrak{L}^0$.

We define the global interface corrector $R^p$ by adding the contributions of all patches. Let $\mathring{R}^p_P$ the extension by zero of $R^p_P$ outside $P$ and set

$$R^p u = \sum_{P \in \mathfrak{P}^p} \mathring{R}^p_P u, \quad \text{for } u \in V^p_{\mathrm{nod}}(\mathfrak{O}^p). \tag{4.12}$$

We recall that, by virtue of $(\mathrm{P}_1)$, the patches $P \in \mathfrak{P}^p$ do not contain any terminal element $K \in \mathfrak{T}^p$. Therefore $R^p u\big|_K = 0$ for all $K \in \mathfrak{T}^p$.

Finally we apply the corrector $R^p$ to obtain $\mathcal{I}^p$: If $(\mathrm{I}_1)$-$(\mathrm{I}_5)$ are satisfied for the integer $d \ge 0$, then we set for any $p \ge 3$ and $u \in \mathscr{C}^\infty(\overline{\Omega}^p)$ defined on $\Omega$:

$$\mathcal{I}^p u = J^p u - R^p J^p u. \tag{4.13}$$

Note that according to $(\mathrm{I}_4)$, $J^p u \in V^p_{\mathrm{nod}}(\mathfrak{O}^p)$, so that (4.13) is well defined. Since the discrete spaces satisfy the inclusion $V^p_K \subset W^p_K$, the interpolant $\mathcal{I}^p$ takes its values in the space

$$W^p = \big\{u \in \mathsf{L}^2(\Omega) \mid \quad \forall K \in \mathfrak{M}^p, \quad u\big|_K \in W^p_K\big\}. \tag{4.14}$$

**Lemma 4.8** *If assumptions* $(\mathrm{I}_1)$-$(\mathrm{I}_5)$ *hold with* $d$*, then the interpolant* $\mathcal{I}^p$ *takes its values in the space* $W^p_{(d)} := W^p \cap \mathscr{C}^d(\overline{\Omega})$.

PROOF. Let $u$ belong to $\mathscr{C}^\infty(\overline{\Omega}^p)$ and $v = \mathcal{I}^p u$. It suffices to prove that for all edges $a$ of any element $K \in \mathfrak{M}^p$, the jumps $[\partial^\alpha v]_a$ are zero for all $\alpha$, $|\alpha| \le d$.

• If $a$ is an edge of $K \in \mathfrak{T}^p$, then $J^p u = Z^p u = 0$ and $a$ is outside the support of $R^p$, therefore $v \equiv 0$ in a neighborhood of $a$, therefore its jumps are zero.

• If $a$ is an edge of $K \in \mathfrak{L}^{p-1}$ which does not belong to $\mathfrak{L}^{p-2}$, then, again, $J^p u = Z^p u = 0$. Moreover if $a$ is contained in a patch $P$, it does not belong to the set $A(P)$. Therefore $\partial^\alpha R^p J^p u\big|_a$ is zero for all $\alpha$, $|\alpha| \le d$.

• If $a \in \mathfrak{A}^p$ (the set of the edges of the $K \in \mathfrak{O}^{p-2}$), there exists a unique patch $P \in \mathfrak{P}^p$, such that $a \in A(P)$. Then

$$[\partial^\alpha v]_a = [\partial^\alpha J^p u]_a - [\partial^\alpha R^p J^p u]_a = 0, \quad |\alpha| \le d.$$

since $J^p u\big|_P$ belongs to $V^p_{\mathrm{nod}}(P)$. ∎

The combination of all axioms yields exponential convergence for the interpolant family $\mathcal{I}^p$:

**Theorem 4.9** *Let the family of meshes $\mathfrak{M} = \big(\mathfrak{M}^p\big)$ satisfy assumptions $(\mathrm{M}_0)$-$(\mathrm{M}_3)$, the family of patches $\mathfrak{P} = \big(\mathfrak{P}^p\big)$ assumptions $(\mathrm{P}_1)$-$(\mathrm{P}_3)$, the family of interpolants assumptions $(\mathrm{I}_1)$-$(\mathrm{I}_5)$ for $d \geq 0$. Then for any $\beta' \in \mathbb{R}$, if $u \in \mathsf{A}_{\beta'}(\Omega)$, $\mathcal{I}^p u \in \mathscr{C}^d(\overline{\Omega})$. If, moreover,* (4.5) *holds, We obtain for all $\beta > \beta'$ the exponential convergence rate*

$$\|u - \mathcal{I}^p u\|_{\mathsf{K}^{d+1}_\beta(\Omega)} \leq C\, e^{-bp}, \quad \textit{with} \quad b > 0. \tag{4.15}$$

PROOF. We have

$$\|u - \mathcal{I}^p u\|_{\mathsf{K}^{d+1}_\beta(\Omega)} \leq \|u - J^p u\|_{\mathsf{K}^{d+1}_\beta(\Omega)} + \sum_{P\in\mathfrak{P}^p} \|R^p_P J^p u\|_{\mathsf{K}^{d+1}_\beta(P)}\,.$$

But

$$\|u - J^p u\|_{\mathsf{K}^{d+1}_\beta(\Omega)} \leq \|u - Z^p u\|_{\mathsf{K}^{d+1}_\beta(\widetilde{\mathfrak{T}}^{p-2})} + \|u - \Pi^p u\|_{\mathsf{K}^{d+1}_\beta(\mathfrak{O}^{p-2})}\,.$$

For $\|u - Z^p u\|_{\mathsf{K}^{d+1}_\beta(\widetilde{\mathfrak{T}}^{p-2})}$, we use (4.9). For $\|u - \Pi^p u\|_{\mathsf{K}^{d+1}_\beta(\mathfrak{O}^{p-3})}$ we use (4.6) (recall that $\ell \geq d+1$). And we obtain

$$\|u - J^p u\|_{\mathsf{K}^{d+1}_\beta(\Omega)} \leq C\, e^{-bp}. \tag{4.16}$$

By (4.8)

$$\begin{aligned} \|R^p_P J^p u\|_{\mathsf{K}^{d+1}_\beta(P)} &\leq C \,\inf\big\{\|J^p u - z\|_{\mathsf{K}^{\ell}_\beta(P)} \;\mid\; z \in \mathsf{H}^\ell(U_P)\big\} \\ &\leq C\|J^p u - u\|_{\mathsf{K}^{\ell}_\beta(P)} \quad \text{since } u \in \mathsf{H}^\ell(U_P). \end{aligned}$$

Thanks to assumption $(\mathrm{P}_2)$,

$$\sum_{P\in\mathfrak{P}^p} \|J^p u - u\|_{\mathsf{K}^{\ell}_\beta(P)} \leq C\|J^p u - u\|_{\mathsf{K}^{\ell}_\beta(\Omega)}$$

Using again (4.16), we obtain (4.15). ■

### 4.e Boundary correctors

We finally need (elementwise) boundary correctors to implement Dirichlet boundary conditions in the discrete spaces. For our application to Maxwell, we only need to cancel the first trace $u\big|_{\partial\Omega}$. Thus, we do not address a more general theory.

We assume that we have constructed a family of interpolants $(J^p)$ according to (4.11) and that, moreover, $(\mathrm{I}_1)$-$(\mathrm{I}_5)$ hold for a $d \geq 0$ and that $p \geq 2$. Then, in particular, $J^p$ is zero on $\mathfrak{T}^p \cup \mathfrak{L}^{p-1}$ and takes values in the space $V^p_{\mathrm{nod}}(\mathfrak{O}^p)$.

Let $K \in \mathfrak{O}^p$ with at least one edge $a$ contained in $\partial\Omega$. Let $B$ be the set of remaining edges of $K$. Since in the terminal layers $\mathfrak{T}^p \cup \mathfrak{L}^{p-1}$ the interpolant is already zero, no correction is needed there. For $K \in \mathfrak{O}^{p-1}$, let $V^p_{a;\,\mathrm{nod}}(K)$ be the image under $J^p_K$ of the space $\{v \in \mathscr{C}^\infty(\overline{K})\,;\; v\big|_a = 0\}$. The next definition is in the same spirit (but simpler) as the definition of the interface correctors.

**Definition 4.10** A boundary corrector for the interpolant $J_K^p$ is an operator $B_K^p:\ V^p_{a;\,\mathrm{nod}}(K)\to W_K^p$ satisfying the algebraic condition $(\mathrm{I}_6)$ and the stability condition $(\mathrm{I}_7)$:

$(\mathrm{I}_6)$ For all $v\in V^p_{a;\,\mathrm{nod}}(K)$, the function $w:=B_K^p v$ satisfies

$$w\big|_a = v\big|_a \qquad \text{and} \qquad \forall b\in B,\quad \forall\alpha,\ |\alpha|\le d,\quad \partial^\alpha w\big|_b = 0.$$

$(\mathrm{I}_7)$ With $\breve B_K^p = H_K^* B_K^p (H_K^*)^{-1}$, there hold the uniform estimates

$$\|\breve B_K^p \breve v\|_{\mathsf{H}^{d+1}(\breve K)} \le C\ \inf\big\{\|\breve v-\breve z\|_{\mathsf{H}^\ell(\breve K)}\ \mid\ \ \breve z\in\mathsf{H}^\ell(\breve K)\ \text{ with }\ \breve z\big|_{\breve a}=0\big\}. \qquad \blacksquare$$

Note that the conditions on the edges $b\in B$ ensure that the extension by zero $\mathring{B}_K^p$ of $B_K^p$ defines an operator which takes its values in $\mathscr{C}^d(\overline{\Omega})$. Note also that, since $a$ is contained in $\partial\Omega$, so does not belong to any active set $A(P)$, the interface correctors $R_P^p$ never modify the traces on $a$.

We obtain again a similar statement to Proposition 4.5:

**Proposition 4.11** *Under the above hypotheses for any $\beta\in\mathbb{R}$ and $v\in V^p_{a;\,\mathrm{nod}}(K)$, we have the estimate*

$$\|B_K^p v\|_{\mathsf{K}^{d+1}_\beta(K)} \le C\ \inf\big\{\|v-z\|_{\mathsf{K}^\ell_\beta(K)}\ \mid\ \ z\in\mathsf{H}^\ell(K)\ \textit{ with }\ z\big|_a=0\big\}. \tag{4.17}$$

# 5 Exponential convergence

We are going to list the properties required the $hp$-subspaces $\mathfrak{X}^p$ so that the Galerkin solution $\mathbf{u}_p$ to the discrete problem (1.5) converges exponentially to the solution $\mathbf{u}$ of the Maxwell problem (1.3) (or equivalently (1.1)). Throughout, an admissible weight $\boldsymbol{\gamma}$ (i.e. such that (1.4) holds) for the weighted regularization is fixed.

In the next section, we give three classes of concrete constructions for such discrete spaces, based on different chains of discrete elementary subspaces and interpolants satisfying the conditions $(\mathrm{I}_1)$-$(\mathrm{I}_7)$. All examples are such that $N:=\dim\mathfrak{X}^p=\mathscr{O}\big(p^3\big)$.

Let for each integer $p\ge 2$ the two chains of elementary subspaces and interpolants

$$\begin{cases} V^p_{(0),K},\ \Pi^p_{(0),K},\ W^p_{(0),K},\ Z^p_{(0),K},\ R^p_{(0),P},\ B^p_{(0),K} & \text{satisfy } (\mathrm{I}_1)\text{-}(\mathrm{I}_7) \text{ with (4.5) for } d=0\\ V^p_{(1),K},\ \Pi^p_{(1),K},\ W^p_{(1),K},\ Z^p_{(1),K},\ R^p_{(1),P},\ B^p_{(1),K} & \text{satisfy } (\mathrm{I}_1)\text{-}(\mathrm{I}_7) \text{ with (4.5) for } d=1\end{cases} \tag{5.1}$$

such that for all $K\in\mathfrak{O}^p$

$$\mathbf{grad}\, W^p_{(1),K}\subset W^p_{(0),K}\times W^p_{(0),K}. \tag{5.2}$$

Then we set

$$\mathfrak{X}^p=\big\{\mathbf{v}\in\mathsf{X}^{\boldsymbol{\gamma}}_N\ \mid\quad \forall K\in\mathfrak{T}^p,\ \ \mathbf{v}\big|_K=0,\quad \forall K\in\mathfrak{O}^p,\ \ \mathbf{v}\big|_K\in W^p_{(0),K}\times W^p_{(0),K}\big\}. \tag{5.3}$$

**Remark 5.1** *(i)* $\mathfrak{X}^p$ is equivalently defined as the subspace of the $\mathbf{v}$ in $\prod_{K\in\mathfrak{O}^p} W^p_{(0),K} \times W^p_{(0),K}$ which are zero on $\mathfrak{T}^p$, *continuous on* $\overline{\Omega}$ and satisfy $\mathbf{v}\times\mathbf{n} = 0$ on $\partial\Omega$.
*(ii)* Let $\Phi^p$ be the set of the $\varphi \in \mathsf{H}^1_0(\Omega)$ in $\prod_{K\in\mathfrak{O}^p} W^p_{(1),K}$ which are zero on $\mathfrak{T}^p$ and $\mathscr{C}^1$ on $\overline{\Omega}$. Then conditions (5.2) and (5.3) yield

$$\mathbf{grad}\,\Phi^p \subset \mathfrak{X}^p. \tag{5.4}$$

■

**Theorem 5.2** *Let* $\mathbf{f}\in\mathbf{A}_0(\Omega)$ *with* $\operatorname{div}\mathbf{f} = 0$. *Let the family of discrete spaces* $(\mathfrak{X}^p)$ *be defined according to* (5.1)-(5.3), *where the underlying family of meshes* $\mathfrak{M} = (\mathfrak{M}^p)$ *satisfies conditions* $(\mathrm{M}_0)$-$(\mathrm{M}_3)$. *Then* $N = \dim\mathfrak{X}^p = O(p^3)$ *as* $p\to\infty$ *and the* $hp$*-FE approximations* $\mathbf{u}_p$ *defined in* (1.5) *converge exponentially to the solution* $\mathbf{u}$ *of problem* (1.3), *i.e. there are* $b, b', C > 0$ *independent of* $p$ *such that*

$$\|\mathbf{u}-\mathbf{u}_p\|_{\mathsf{X}^\gamma_N(\Omega)} \le C\,e^{-b'p} = Ce^{-b\sqrt[3]{N}} \quad \textit{as} \quad p\to\infty. \tag{5.5}$$

PROOF. Corollary 2.8 gives the splitting $\mathbf{u} = \mathbf{grad}\,\varphi + \mathbf{w}$ (with $\varphi\in\mathsf{H}^1_0(\Omega)$ and $\mathbf{w}\in\mathsf{H}^1_N(\Omega)$, – let us recall that $\mathsf{H}^1_N(\Omega)$ is the subspace of $\mathsf{H}^1(\Omega)^2$ with zero tangential trace) together with the weighted analytic regularity (2.14)

$$\varphi\in\mathsf{A}_{\boldsymbol{\gamma}-2-\delta}(\Omega) \quad \text{and} \quad \mathbf{w}\in\mathbf{A}_{-1-\delta}(\Omega), \quad \text{with} \quad \delta>0.$$

We have the energy estimate (2.15)

$$\|\mathbf{u}\|_{\mathsf{X}^\gamma_N(\Omega)} \le \|\varphi\|_{\mathsf{K}^2_{\gamma-2}(\Omega)} + \|\mathbf{w}\|_{\mathbf{K}^1_{-1}(\Omega)}\,.$$

Therefore for any $\mathbf{u}_p\in\mathfrak{X}^p$ in the form $\mathbf{grad}\,\varphi_p + \mathbf{w}_p$ with $\varphi_p\in\Phi^p$ and $\mathbf{w}_p\in\mathfrak{X}^p$ we have

$$\|\mathbf{u}-\mathbf{u}_p\|_{\mathsf{X}^\gamma_N(\Omega)} \le \|\varphi-\varphi_p\|_{\mathsf{K}^2_{\gamma-2}(\Omega)} + \|\mathbf{w}-\mathbf{w}_p\|_{\mathbf{K}^1_{-1}(\Omega)}\,. \tag{5.6}$$

Thus, we are going to choose $\varphi_p\in\Phi^p$ as an interpolant of $\varphi$ and $\mathbf{w}_p$ as an interpolant of $\mathbf{w}$. Using (5.4), we have that $\mathbf{grad}\,\varphi_p + \mathbf{w}_p$ belongs to $\mathfrak{X}^p$ and is an interpolant for $\mathbf{u}$.
Defining the boundary corrector

$$B^p_{(1)} = \prod_{a\,\in\,\partial\Omega\,\cap\,\mathfrak{O}^{p-1}} \mathring{B}^p_{(1),K}$$

with $\mathring{B}^p_{(1),K}$ the extension by zero of $B^p_{(1),K}$, we modify the interpolant (4.13)

$$\widetilde{\mathcal{I}}^p_{(1)}\varphi = J^p_{(1)}\varphi - R^p_{(1)}J^p_{(1)}\varphi - B^p_{(1)}J^p_{(1)}\varphi.$$

Thanks to conditions $(\mathrm{I}_6)$-$(\mathrm{I}_7)$ this interpolant acts from $\mathsf{A}_{\boldsymbol{\gamma}-2-\delta}(\Omega)\cap\mathsf{H}^1_0(\Omega)$ into $\Phi^p$ and satisfies the same exponential estimates as in Theorem 4.9. Therefore we obtain for $\beta = \boldsymbol{\gamma}-2$ and $\beta' = \boldsymbol{\gamma}-2-\delta$ the exponential estimate (note that $d+1=2$)

$$\|\varphi-\widetilde{\mathcal{I}}^p_{(1)}\varphi\|_{\mathsf{K}^2_{\gamma-2}(\Omega)} \le C\,e^{-bp}, \quad \text{with} \quad b>0. \tag{5.7}$$

For any edge $a \in \partial\Omega \cap \mathfrak{O}^{p-1}$, let $\boldsymbol{\tau}_a$ be a tangential unit vector to $a$. Then we can define the tangential boundary corrector

$$B^p_{(0)}\mathbf{w} = \prod_{a\in\partial\Omega\cap\mathfrak{O}^{p-1}} \overset{\circ}{B}{}^p_{(0),K}(\mathbf{w}\cdot\boldsymbol{\tau}_a)$$

and we modify the interpolant (4.13)

$$\widetilde{\mathcal{I}}^p_{(0)}\mathbf{w} = J^p_{(0)}\mathbf{w} - R^p_{(0)}J^p_{(0)}\mathbf{w} - R^p_{(0)}B^p_{(0)}J^p_{(0)}\mathbf{w}.$$

This interpolant acts from $\mathbf{A}_{-1-\delta}(\Omega)\cap\mathsf{H}^1_N(\Omega)$ into $\mathfrak{X}^p$. Again by a modification of Theorem 4.9, we obtain for $\beta = -1$ and $\beta' = -1-\delta$ the exponential estimate (now $d+1=1$)

$$\|\mathbf{w} - \widetilde{\mathcal{I}}^p_{(0)}\mathbf{w}\|_{\mathbf{K}^1_{-1}(\Omega)} \le C\,e^{-bp}, \quad \text{with} \quad b>0. \tag{5.8}$$

The inequalities (5.6)-(5.8) yield (5.5). ∎

## 6 Three concrete $hp$-Element Families

Here we present the two chains of elementary subspaces and interpolants according to requirements (5.1)-(5.2) for three different families of $hp$-elements for which we verify the conditions of the preceding convergence analysis. The element families considered consist of **(a)** rectangular elements on geometric meshes with hanging nodes, or **(b)** triangular elements on regular geometric meshes, or **(c)** bilinearly mapped quadrilaterals on geometric meshes.

In each case, a $\mathscr{C}^1$ conforming $hp$-interpolant will be be constructed on the geometric mesh under consideration, implying exponential convergence of the corresponding $\mathscr{C}^0$ $hp$-FEM for the weighted regularization of Maxwell's equations. Our $hp$ interpolants may also be of interest in approximation of plate and shell problems. Further, our construction of $\mathscr{C}^1$-conforming $hp$ interpolants is flexible: $\mathscr{C}^1$ conforming interpolants on other geometric mesh families, e.g. on combinations of affine quadrilaterals and triangles, with exponential convergence estimates are readily constructed with the tools developed here.

Thus, in this section we are going to prove that the generic families of elements quoted above satisfy conditions (5.1)-(5.2) with suitable choices of elemental polynomial spaces. All our interpolants are based on the basic tensorial interpolants $\Pi^p_d$ of the reference square constructed and studied in §8.b. We note in particular that approximation estimates (8.20) are compatible with the exponential bound (4.5) of the $\Psi_{p,k}$.

For nodal and trace liftings we will use the following family of polynomials on the standard interval $I=(0,1)$: Let $d\ge 0$ and $i$, $0\le i\le d$. There are (unique) functions $\chi_{d,i}\in\mathbb{P}^{2d+1}$ such that for all $j$, $0\le j\le d$ there holds

$$\chi^{(j)}_{d,i}(0) = \delta_{ij} \quad \text{and} \quad \chi^{(j)}_{d,i}(1) = 0\,. \tag{6.1}$$

Generic trace lifting from one edge of a reference square or triangle are stated in §8.c and 8.d. We do more specific constructions here for the interface correctors.

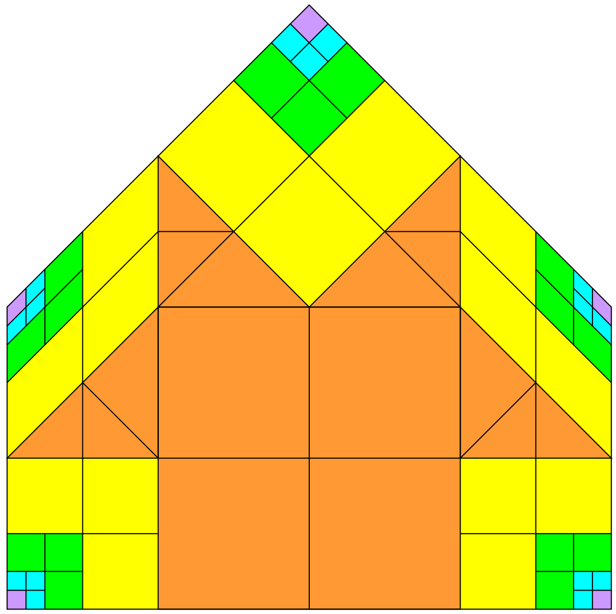

Figure 4: Composite mesh with hanging nodes

## 6.a Affine quadrilaterals (rectangles) with hanging nodes

Here we consider affine quadrilaterals in the following restrictive sense: *There exists a global affine mapping which transforms the whole mesh into a rectangular mesh with hanging nodes.* Thus the directional derivatives $\partial_1$ and $\partial_2$ are the derivatives along the axes of this global affine mapping, and, from now on, we work directly on the rectangular mesh. We consider rectangular elements $K$ with at most one hanging node per side. The reference element is the square $\widehat{K} = (-1,1)^2$.

It is not hard to see that our analysis allows to combine several meshes of this type, plus additional triangular and quadrilateral elements, under the condition that the matching between different meshes is done in the unrefined regions, see Figure 4. The geometric meshes investigated in [2] are similar.

### *6.a.(i)* ***Primary interpolants***

The elemental spaces $V^p_{(d),K}$ for $d = 0,\ 1$ are transported from the same tensor space $\mathbb{Q}^p$ on the reference square $\widehat{K}$.

The interpolants $\Pi^p_{(d),K}$ are transported from the interpolants $\Pi^p_d$ on $\widehat{K}$ constructed in Theorem 8.5. Since in that case, $\widehat{\Pi}^p_K$ coincides with $\Pi^p_d$, Theorem 8.5 gives immediately property $(\mathrm{I}_1)$ combined with the estimate (4.5) of the $\Psi_{p,k}$.

### *6.a.(ii)* ***Interface correctors***

We now verify Properties $(\mathrm{I}_2)$, $(\mathrm{I}_3)$ in Definition 4.3, and construct the interface correctors of order $d \geq 0$ on the patches $P \in \mathfrak{P}^p$: the discrete spaces $W^p_{(d),K} = V^p_{(d),K}$ here. We are going to construct the lifting $R^p_{(d),P}$.

We note that, by construction of $\Pi^p_{(d),K}$, we have for any subset $P$ of the mesh $\mathfrak{M}^p$

$$V^p_{\text{nod}}(P) = \big\{ w \mid \quad \forall K \in P,\ w\big|_K \in V^p_K, \quad \forall K, K' \in P,\ \forall N \in \overline{K} \cap \overline{K}', \\ \partial_1^j \partial_2^k w\big|_K (N) = \partial_1^j \partial_2^k w\big|_{K'} (N),\ 0 \le j,k \le d \big\}. \tag{6.2}$$

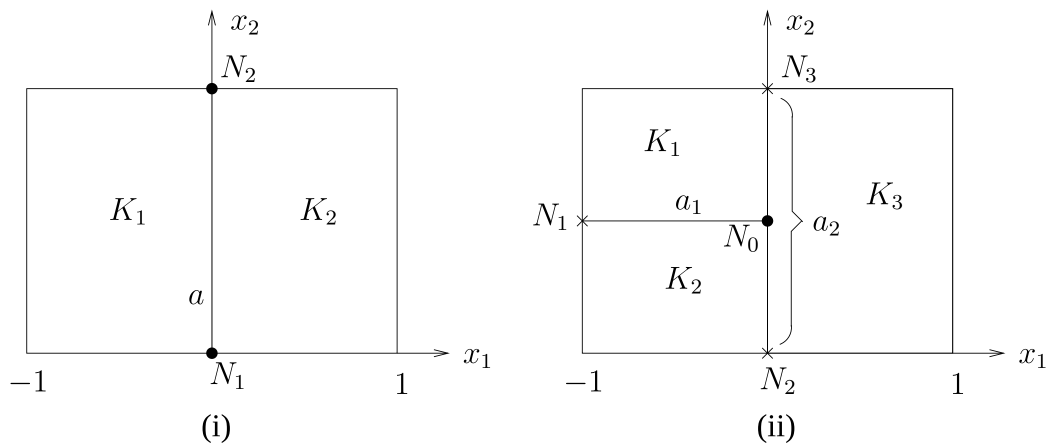


Figure 5: Two patches in geometric mesh with hanging nodes and notation

It is sufficient to consider two types of patches shown in Fig. 5. Denote by $a_{j,k}$ the edges of element $K_j, k = 1, 2, 3, 4$.

- *Patch $P = (K_1, K_2)$ of two elements*

The two elements $K_1$ and $K_2$ share an entire, active edge $a \in A$, say, $a = a_{1,1} = a_{2,1}$. The inactive edges (i.e. where the lifting of the jumps across $a$ will have no influence) are $b \in B = \{a_{j,k} : j = 1, 2, k = 2, 3, 4\}$. Denote by $N_1, N_2$ the endpoints of $a$, $\partial a = \{N_1, N_2\}$. For any $V^{(0)} \in V^p_{\text{nod}}(P)$, the tangential derivatives $\partial^\ell_\tau$ of the normal jumps $[\partial^k_n V^{(0)}]_a$ satisfy by (8.19) the nodal compatibility conditions at the nodes $N_j$, $j = 1, 2$

$$\partial^\ell_\tau \big[\partial^k_n V^{(0)}\big]_a (N_j) = 0 \quad \forall k, \ell = 0, \ldots, d\,. \tag{6.3}$$

To remove the normal jumps of $V^{(0)}$ across $a$, we make use directly of Proposition 8.6: there exist polynomials $\Phi_i(x_1, x_2)$ such that

$$\partial^\ell_n \Phi_i\big|_a = \delta_{i\ell} \big[\partial^i_n V^{(0)}\big]_a \quad \text{and} \quad \partial^\ell_n \Phi_i\big|_{\partial K_1 \setminus a} = 0, \qquad 0 \le \ell \le d\,.$$

The lifting $R^p_{d,P}$ of $V^{(0)} \in V^p_{\text{nod}}(P)$ is then given by

$$R^p_{d,P}\, V^{(0)} := \left\{ \begin{array}{cl} -\displaystyle\sum_{i=0}^{d} \Phi_i(x_1, x_2) & \text{in } K_1 \\ 0 & \text{in } K_2 \end{array} \right\} \tag{6.4}$$

and the corrected function

$$V = V^{(0)} + R^p_{d,P}\, V^{(0)} \in V^p(K_i),\ i = 1, 2 \tag{6.5}$$

satisfies for $k, \ell = 0, \ldots, d$:

$$\partial_\tau^\ell \big[\partial_n^k V\big]_a \equiv 0 \quad \text{on} \quad a = \overline{K}_1 \cap \overline{K}_2 \,.$$

Hence $V \in \mathscr{C}^d(\overline{K}_1 \cup \overline{K}_2)$ and $R^p_{d,P}$ satisfies $(\mathrm{I}_2)$, $(\mathrm{I}_3)$ for $P = (K_1, K_2)$.

- *Patch $P = (K_1, K_2, K_3)$ of three rectangles*

The three rectangles have two edges in common: an edge $a_1 = \overline{K}_1 \cap \overline{K}_2 \in A$, i.e. $a_1 = a_{1,1} = a_{2,1}$ and another edge $a_2 \in A$ shared by all three elements, say

$$a_2 = a_{3,2} \quad \text{and} \quad a_{1,2} \subset a_2, \quad a_{2,2} \subset a_2 \,,$$

*cf* Fig. 5 (ii). The node $\bullet$ in Fig. 5 (ii) is hanging: $N_0 = \overline{a}_1 \cap \overline{a}_{1,2} \cap \overline{a}_{2,2}$. By $N_1, N_2, N_3$, we denote the ends of edges $a_1, a_2 \in A$ as shown in Fig. 5 (ii).

For $u \in C^\infty(P)$, define $V^{(0)} \in V^p_{\mathrm{nod}}(P)$ by $V^{(0)}|_{K_i} = \Pi^p_{d,K_i} u$ where $d \geq 0$ is a fixed degree of conformity. Then $V^{(0)}$ satisfies in each $K_i \in P$ the nodal exactness (8.19) of order $d$ and the estimates

$$\|u - V^{(0)}\|_{\mathsf{H}^\ell(P)} \leq \Psi_{p,k} \, \|u\|_{\mathsf{H}^k(P)} \qquad 0 \leq \ell < k \leq p \tag{6.6}$$

with $\Psi_{p,k}$ as in (4.5), where Sobolev norms over $P$ are broken. We construct the lifting of class $\mathscr{C}^d$ for $V^{(0)}$ on $P$ in three steps and refer to Figure 1, (ii).

*(a) Lifting on edge $a_1$.* The jumps $[\partial_n^k V^{(0)}]_{a_1}$ across edge $a_1$ satisfy for $k = 0, \ldots, d$ the nodal compatibility conditions (6.3) at the nodes $N_i$, $i = 0, 1$. For sufficiently large $p$, $[\partial_n^k V^{(0)}]_{a_1}$ may be lifted as in case (i) to $K_1$ by a trace-lifting $R_d^{(1)} V^{(0)} \in \mathbb{Q}^p(K_1)$ such that

$$V^{(1)} := \begin{cases} V^{(0)} + R_d^{(1)} V^{(0)}, & \text{in } K_1, K_2 \\ V^{(0)} & \text{in } K_3 \end{cases} \tag{6.7}$$

is in $\mathscr{C}^d(\overline{K}_1 \cup \overline{K}_2)$, and such that the values $\partial^\alpha V^{(0)}$, $0 \leq \alpha_1, \alpha_2 \leq d$, in the nodes of $P$ are not changed.

*(b) Compatibility at $N_0$.* It will be achieved by modifications of $V^{(1)}$ in $K_1, K_2$ as follows. By step (a), $[\partial_2^\ell V^{(1)}]_{a_1} \equiv 0$, $\ell = 0, \ldots, d$. Therefore the jumps $J_{k\ell} := \partial_2^\ell [\partial_1^k V^{(1)}]_{a_2}(N_0)$ of $V^{(1)}$ across edge $a_2$ in hanging node $N_0$ are well defined for $k, \ell = 0, \ldots, d$. For the lifting of $J_{k\ell}$, we use the polynomials $\chi_{i,d} \in \mathbb{P}^{2d+1}$ introduced in (6.1) and set

$$R_d^{(2)} V^{(1)} := -\sum_{k,\ell=0}^{d} J_{k\ell} \begin{cases} (-1)^k \chi_{k,d}(-x_1)\chi_{\ell,d}(x_2) & \text{in } K_1 \\ (-1)^{k+\ell} \chi_{k,d}(-x_1)\chi_{\ell,d}(-x_2) & \text{in } K_2 \end{cases},$$

$$R_d^{(2)} V^{(1)} := 0 \ \text{ in } \ K_3 \,.$$

Then by (6.1)

$$\partial_1^i \, \partial_2^j (R_d^{(2)} V^{(1)}|_{K_1})(N_0) = -\sum_{k,\ell=0}^{d} J_{k,\ell} (-1)^{k+i} \, \delta_{ik} \delta_{j\ell} = -J_{ij}$$

and $J_{ij}$ is likewise attained by $(R_d^{(2)}\,V^{(1)}|_{K_2})(N_0)$. Moreover, for $j = 0,\dots,d$,

$$\partial_2^j(R_d^{(2)}\,V^{(1)})\big|_{\substack{K_1\\x_2=0}} = -\sum_{k,\ell=0}^{d} J_{k\ell}(-1)^k\,\chi_{k,d}(-x_1) = \partial_2^j(R_d^{(2)}\,V^{(1)})\big|_{\substack{K_2\\x_2=0}}\,,$$

i.e.

$$\big[\partial_2^j\,R_d^{(2)}\,V^{(1)}\big]_{a_1} \equiv 0 \;\text{ on }\; a_1, j = 0,\dots,d\,.$$

Therefore $R_d^{(2)}\,V^{(1)} \in \mathscr{C}^d(\overline{K}_1 \cup \overline{K}_2)$ and $\partial^\alpha\,R_d^{(2)}\,V^{(1)} = 0$ for $0 \le \alpha_1, \alpha_2 \le d$ in all nodes of $K_1, K_2$ except $N_0$. Define

$$V^{(2)} := \begin{cases} V^{(1)} + R_d^{(2)}\,V^{(1)} & \text{in } K_1 \cup K_2\,, \\ V^{(1)} & \text{in } K_3\,. \end{cases} \tag{6.8}$$

Then $V^{(2)} \in \mathscr{C}^d(\overline{K}_1 \cup \overline{K}_2)$, $[V^{(2)}]_{a_2} \in \mathbb{P}^p(a_{i,2})$ for $i = 1, 2$ and for $0 \le k, \ell \le d$ it holds

$$0 = [\partial_1^k\,\partial_2^\ell(u - V^{(2)})]_{a_2}(N_i) = -[\partial_1^k\,\partial_2^\ell\,V^{(2)}]_{a_2}(N_i),\; i = 0, 2, 3\,. \tag{6.9}$$

*(c) Lifting on edges* $a_{i,2}$. Therefore $[\partial_1^k\,V^{(2)}]_{a_2}$ is a polynomial of degree $p$ on the pieces $a_{1,2}$, $a_{2,2}$ of $a_2$ with $\partial_2^\ell[\partial_1^k\,V^{(2)}]_{a_2}(N_i) = 0$, $i = 0, 2, 3$, for $k, \ell = 0, 1, \dots, d$. We may therefore lift $[\partial_1^k\,V^{(2)}]_{a_{i,2}}$ separately into $\mathbb{Q}^p(K_i)$, $i = 1, 2$, such that $V^{(2)}$ and its derivatives up to order $d$ remain unchanged on $\partial K_i \backslash a_{i,2}$: call the lifting $R_d^{(3)}\,V^{(2)}$ and set

$$V^{(3)} := \begin{cases} V^{(2)} + R_d^{(3)}\,V^{(2)} & \text{in } K_1 \cup K_2\,, \\ V^{(2)} & \text{in } K_3\,. \end{cases} \tag{6.10}$$

Then $V^{(3)} \in \mathbb{Q}^p(K_i)$, $i = 1, 2, 3$, $V^{(3)} \in \mathscr{C}^d(\overline{U}_P)$ and $V^{(3)}$ is given by

$$V^{(3)} := \begin{cases} (I + R_d^{(3)})(I + R_d^{(2)})(I + R_d^{(1)})\,V^{(0)} & \text{in } K_1 \cup K_2\,, \\ V^{(0)} & \text{in } K_3 \end{cases} \tag{6.11}$$

and the interface corrector $R_{d,P}^p$, given by

$$R_{d,P}^p\,V^{(0)} := \begin{cases} R_d^{(1)}\,V^{(0)} + R_d^{(2)}\,V^{(1)} + R_d^{(3)}\,V^{(2)} & \text{in } K_1 \cup K_2\,, \\ 0 & \text{in } K_3 \end{cases}$$

satisfies $(\mathrm{I}_2)$.

To verify $(\mathrm{I}_3)$, we observe that for any edge $a$ in $P$, we have the trace inequality

$$\|\varphi|_a\|_{\mathsf{L}^2(a)} \le C\,\|\varphi\|_{\mathsf{H}^1(P)}\,. \tag{6.12}$$

Since $R_d^{(1)} V^{(0)}$ depends only on $[V^{(0)}]_{a_1}$, we have $R_d^{(1)} V^{(0)} = R_d^{(1)}(V^{(0)} - z)$ for any $z \in C^\infty(\overline{U}_P)$, and get for any $k \geq d+1$ with (6.12)

$$\begin{aligned} \|R_d^{(1)} V^{(0)}\|_{\mathsf{H}^k(P)} &\leq C \sum_{i=0}^{d} \|[\partial_n^i V^{(0)}]\|_{\mathsf{H}^{k-i}(a_1)} \\ &= C \sum_{i=0}^{d} \|[\partial_n^i V^{(0)} - z]\|_{\mathsf{H}^{k-i}(a_1)} \\ &\leq C \|V^{(0)} - z\|_{\mathsf{H}^{k+1}(P)} . \end{aligned} \tag{6.13}$$

Likewise,

$$\|R_d^{(3)} V^{(2)}\|_{\mathsf{H}^k(P)} \leq C \|V^{(2)} - z\|_{\mathsf{H}^{k+1}(P)} . \tag{6.14}$$

For $R_d^{(2)}$, we observe that e.g. on $K_1$ for any $z \in C^\infty(\overline{U}_P)$

$$\begin{aligned} \|R_d^{(2)} V^{(1)}\|_{\mathsf{H}^k(K_1)} &\leq C \sum_{i,j=0}^{d} |J_{ij}| \\ &\leq C \|V^{(1)} - z\|_{\mathsf{H}^{2d+2}(P)} \\ &= C \|V^{(0)} + R_d^{(1)} V^{(0)} - z\|_{\mathsf{H}^{2d+2}(P)} \\ &\leq C \left\{ \|z - V^{(0)}\|_{\mathsf{H}^{2d+2}(P)} + \|R^{(1)} V^{(0)}\|_{\mathsf{H}^{2d+2}(P)} \right\} \\ &\overset{(6.13)}{\leq} C \|z - V^{(0)}\|_{\mathsf{H}^{2d+3}(P)} , \end{aligned}$$

hence, for any $z \in C^\infty(\overline{U}_P)$, $0 \leq k \leq d+1$,

$$\|R_d^{(2)} V^{(1)}\|_{\mathsf{H}^k(P)} \leq C \|z - V^{(0)}\|_{\mathsf{H}^{2d+3}(P)} . \tag{6.15}$$

With the definition of $V^{(2)}$, we get

$$\begin{aligned} \|V^{(2)} - z\|_{\mathsf{H}^{k+1}(P)} &= \|V^{(1)} + R_d^{(2)} V^{(1)} - z\|_{\mathsf{H}^{k+1}(P)} \\ &\leq \|V^{(0)} - z\|_{\mathsf{H}^{k+1}(P)} + \|R_d^{(1)} V^{(0)}\|_{\mathsf{H}^{k+1}(P)} + \|R_d^{(2)} V^{(1)}\|_{\mathsf{H}^{k+1}(P)} . \end{aligned}$$

Combining this with (6.13) - (6.15), we get

$$\|R_{d,P}^{p} V^{(0)}\|_{\mathsf{H}^{d+1}(P)} \leq C \|V^{(0)} - z\|_{\mathsf{H}^{2d+3}(P)} . \tag{6.16}$$

A density argument and a scaling imply $(\mathrm{I}_3)$.

#### 6.a.(iii) *Corner correctors*

We have to define the corner interpolant $Z^p_K$, so to satisfy conditions $(\mathrm{I}_4)$ and $(\mathrm{I}_5)$. By definition, $Z^p_K$ is zero for any $K$ in the terminal layers $\mathfrak{T}^p \cup \mathfrak{L}^{p-1}$. It remains to define $Z^p_K$ for any $K \in \mathfrak{L}^{p-2}$, so that the extension of $Z^p$ by $\Pi^p$ in $\mathfrak{O}^{p-2}$ takes its values in $V^p_{\mathrm{nod}}(\mathfrak{O}^p)$. According to (6.2), this means that the nodal values $\partial_1^j\partial_2^k w(N)$, $0 \le j,k \le d$, have to be uniquely defined for any $\overline{K}$ containing $N$.

Let $u \in \mathscr{C}^\infty(\overline{\Omega}^p)$ and let $K \in \mathfrak{L}^{p-2}$. If $\overline{K}$ does not intersect any $\overline{K}'$ with $K' \in \mathfrak{L}^{p-3}$, we set $Z^p_K u = 0$. If not, let $\mathscr{N}$ be those vertices $N$ which $K$ shares with elements $K' \in \mathfrak{L}^{p-3}$, and let $\mathscr{M}$ be the set of the remaining nodes of $K$. For any $p > 2d+1$, define $w := Z^p_K u$ as unique Hermite interpolant in $\mathbb{Q}^{2d+1}(K)$ such that

$$\forall N \in \mathscr{N},\;\; \partial_1^j\partial_2^k w(N) = \partial_1^j\partial_2^k u(N) \quad\text{and}\quad \forall N \in \mathscr{M},\;\; \partial_1^j\partial_2^k w(N) = 0,\;\; 0 \le j,k, \le d.$$

The stability of $Z^p_K$ in $\mathsf{H}^{2d+2}(\widehat{K})$ is obvious.

We then define $J^p u$ according to $(\mathrm{I}_4)$ by extending $\Pi^p$ by $Z^p$ in $\mathfrak{L}^{p-2} \cap \mathfrak{L}^{p-1} \cap \mathfrak{T}^p$. Since condition $(\mathrm{M}_0)$ gives the separation between layers $\mathfrak{L}^{p-3}$ and $\mathfrak{L}^{p-1}$, the whole construction above yields an element $w = J^p u$ which belongs to $V^p_{\mathrm{nod}}(\mathfrak{O}^p)$ as required.

#### 6.a.(iv) *Boundary correctors*

Let $a$ be an edge contained in $\partial\Omega$ and let $K$ be the element containing $a$. We have to define the lifting operator $B^p_K$ satisfying $(\mathrm{I}_6)$ and $(\mathrm{I}_7)$. If $K$ belongs to $\mathfrak{T}^p \cup \mathfrak{L}^{p-1}$, nothing is to be done, since the interpolant $J^p_K$ coincides with $Z^p_K$ which vanishes there. Let now $K$ belong to $\mathfrak{O}^{p-1}$ and $u \in \mathscr{C}^\infty(\overline{\Omega}^p) \cap \mathsf{H}^1_0(\Omega)$. We set $w := J^p_K u$ and $\varphi := w\big|_a$. Let $N_1$ and $N_2$ be the endpoints of edge $a$.

If $K$ belongs to $\mathfrak{O}^{p-2}$, $J^p_K = \Pi^p_K$. Therefore, by (8.19), for $i = 1,2$:

$$\varphi^{(j)}(N_i) = \partial_\tau^j u(N_i), \quad j = 0,\dots,d. \tag{6.17}$$

Since $u\big|_a \equiv 0$, we find that $\varphi^{(j)}(N_i) = 0$, $j = 0,\dots,d$, which is condition (8.21). Thus, Proposition 8.6 yields $\Phi_0$ defining $B^p_K w$ with suitable trace properties and stability in $\mathsf{H}^{d+1}(\widehat{K})$.

If $K$ belongs to $\mathfrak{L}^{p-2}$, $J^p_K$ is now defined as in the section above. If $N_i$ belongs to $\mathscr{N}$, then (6.17) still holds, therefore $\varphi^{(j)}(N_i) = 0$. If $N_i$ belongs to $\mathscr{M}$, $\partial_\tau^j w(N_i)$ is zero by construction for $j = 0,\dots,d$. Thus we can end the construction as before.

#### 6.a.(v) *Conclusion*

With $W^p_{(d),K} = \mathbb{Q}^p(K)$ for $d = 0,\,1$, we check immediately property (5.2), i.e. the embedding of $\mathbf{grad}\, W^p_{(1),K}$ into $W^p_{(0),K} \times W^p_{(0),K}$. This ends the verification of all our axioms in the case of rectangles with hanging nodes.

### 6.b Conforming parallelograms and triangles

We consider meshes $\mathfrak{M}^p$ formed with parallelograms and triangles, affine equivalent to the reference elements, and we assume that they are conforming, that is, that the intersection of two distinct elements is either empty, one node, or an entire edge. This case was already considered in [29]. We show here how this case fits our axioms. Moreover the condition we will introduce on triangles is simpler as in *loc. cit.*, and generically always satisfied. We assume that $(\mathfrak{M}^p)$ satisfies $(\mathrm{M}_0) - (\mathrm{M}_3)$.

### *6.b.(i)* ***Primary interpolants***

• *Parallelograms.* Like in the rectangular case, the elemental spaces $V^p_{(d),K}$ for $d = 0, 1$ are transported from the same tensor space $\mathbb{Q}^p$ on the square $\widehat{K}$. Again we take $W^p_{(d),T} = V^p_{(d),T}$.

But now, whereas for $d = 0$ the interpolant $\Pi^p_{(0),K}$ is still transported from the interpolant $\Pi^p_0$ on $\widehat{K}$, for $d = 1$ the interpolant $\Pi^p_{(1),K}$ is transported from the interpolant $\Pi^p_2$ on $\widehat{K}$, ensuring in particular the nodal interpolation for all vertices $N$ of $K$

$$\partial^\alpha(\Pi^p_{(1),K}u)(N) = \partial^\alpha u(N), \quad \forall\alpha,\ |\alpha| \le 2. \tag{6.18}$$

Theorem 8.5 gives immediately property $(\mathrm{I}_1)$ combined with the estimate (4.5) of the $\Psi_{p,k}$.

• *Triangles.* All triangles $T \in \mathfrak{M}^p$ are affine images of the reference element $S^2$. We assume that for each $k \ge 0$ and each $T \in \mathfrak{L}^k$ there is a parallelogram $K_T$ sharing three common corners with $T$ and such that

$$K_T \subset \bigcup_{T' \in \mathfrak{L}^k \cup \mathfrak{L}^{k-1}} \overline{T}'.$$

Note that such an assumption is generically satisfied since the "width" of the layer $\mathfrak{L}^{k-1}$ is larger that the width of $\mathfrak{L}^k$, which is itself larger than the diameter of $T$. In the above assumption, it is of course understood that $K_T$ *does not itself belong to* $\mathfrak{M}^p$. Moreover, it is not assumed that $K_T \setminus T$ belongs to $\mathfrak{M}^p$ (in other words, the fourth node of $K_T$ does not need to be a node of $\mathfrak{M}^p$).

We omit the subscript $T$ on $K_T$ if its relation to triangle $T$ is clear. Then $T = F_T(S^2)$, $K = F_K(I^2)$. As a consequence of assumptions $(\mathrm{M}_1) - (\mathrm{M}_3)$ there exists a fixed integer $M$ such that for all $T \in \mathfrak{O}^p$, the number of elements $T' \in \mathfrak{M}^p$ having a non-empty intersection with $K_T$ is bounded by $M$.

We consider only the cases $d = 0$ and $1$, which is sufficient for our application. The elemental spaces $V^p_{(d),T} = \mathbb{P}^{2p}(T)$ are transported from $\mathbb{P}^{2p}(S^2)$ and we take $W^p_{(d),T} = V^p_{(d),T}$. We define the primary interpolants as

$$\Pi^p_{d,T}u := (\Pi^p_{2d}(u|_K \circ F_K)) \circ F_T^{-1}, \quad d = 0, 1 \tag{6.19}$$

transported from the interpolation operators (8.18) in $I^2$ for orders $0$ and $2$, – Note that we need approximation estimates in $\mathsf{H}^{d+1}$ norm. Transporting $\Pi^p_{d,T}$ back to $S^2$ using $F^*_T$, we find for $\widehat{u} \in C^\infty(\overline{I}^2)$

$$\widehat{\Pi}^p_{d,T}\widehat{u} := \Pi^p_{2d}\widehat{u}|_{S^2} \in \mathbb{Q}^p \subset \mathbb{P}^{2p}(S^2). \tag{6.20}$$

As a consequence of the approximation estimate (8.20), we obtain, instead (4.1)

$$\|\hat{u} - \widehat{\Pi}^p_{d,T}\hat{u}\|_{\mathsf{H}^{d+1}(\widehat{T})} \le \Psi_{p,k}\|\hat{u}\|_{\mathsf{H}^k(\widehat{K})} \tag{6.21}$$

with the $\Psi_{p,k}$ satisfying the exponential bound (4.5). The finite intersection condition above allows to draw the same consequences from (6.21) than from (4.1).

Since the three nodes $N$ of $T$ are nodes of the associated parallelogram $K$, condition (8.19) yields in particular that

$$\Pi^p_{0,T}u(N) = u(N) \quad \text{and} \quad \forall\alpha,\ |\alpha| \le 2, \quad \partial^\alpha\Pi^p_{1,T}u(N) = \partial^\alpha u(N). \tag{6.22}$$

### *6.b.(ii)* ***Interface correctors***

From now on we equally denote by $K$ parallelogram or triangle elements. The patches consist of element pairs $P=(K_1,K_2)$ which share an entire edge $a$, and where $K_1$ and $K_2$ can be both parallelograms, or both triangles, or one of each sort. We agree that if $K_1$ is a parallelogram, so is $K_2$. Thanks to (6.22) the image $V^p_{(d),\mathrm{nod}}(P)$ of $\mathscr{C}^\infty(\overline{U}_P)$ through $\Pi^p_{(d)}(P)$ is

$$V^p_{(d),\mathrm{nod}}(P)=\big\{w\mid\quad w\big|_{K_j}\in V^p_{(d),K_i},\ i=1,2,\quad\forall N\in\overline{K}_1\cap\overline{K}_2,\\ \partial^\alpha w\big|_{K_1}(N)=\partial^\alpha w\big|_{K_2}(N),\ |\alpha|\le 2d\big\}.\tag{6.23}$$

We are going to construct the lifting $R^p_{(d),P}$. We set, for $u\in\mathscr{C}^\infty(\overline{U}_P)$,

$$V^{(1)}|_{K_i}=\Pi^p_{K_i}u,\ i=1,2\,.$$

We detail the proof for $d=1$ (for $d=0$, it is easier). Denote by $n$ the normal to $a$ pointing from $K_1$ into $K_2$. Then for $j=0,\,1$ the normal jumps

$$\varphi_j(s)=\big[\partial^j_n V^{(1)}\big]_a(s)\in\mathbb{P}^{2p}(a),\quad(\in\mathbb{P}^p(a)\text{ if }K_1\text{ parallelogram}),$$

satisfy, by (6.23),

$$\varphi_j^{(i)}(s)=0\qquad s\in\partial a,\quad i=0,1,2-j\,.$$

Applying Proposition 8.7 in $K_1$, with $d=1$, if $K_1$ is a triangle, and Proposition 8.6, if $K_1$ is a parallelogram, gives a lifting $R^{(2)}V^{(1)}$ in $\mathbb{P}^{2p}(K_1)$ if $K_1$ is a triangle, and in $\mathbb{Q}^p(K_1)$ if $K_1$ is a parallelogram, such that

$$V^{(2)}:=\begin{cases}V^{(1)}+R^{(2)}V^{(1)} & \text{in }K_1\\ V^{(1)} & \text{in }K_2\end{cases}\tag{6.24}$$

belongs to $\mathscr{C}^1(\overline{U_P})$.

**Remark 6.1** We obtained here an interpolant with $\mathscr{C}^1$-conformity. It is straightforward, using a higher order vertex correction and Proposition 8.7, to obtain $\mathscr{C}^d$ conforming interpolants for any $d>1$.

### *6.b.(iii)* ***Conclusion***

Like in the rectangular case, the corner corrector is constructed thanks to Lagrange and Hermite interpolants on parallelograms $K$ or triangles $T$: For $d=0$ we simply use the Lagrange interpolants $i_0\otimes i_0:\mathsf{H}^2(K)\to\mathbb{Q}^1(K)$ on the parallelogram and $I^1_T:\mathsf{H}^2(T)\to\mathbb{P}^1(T)$ on the triangle, whereas for $d=1$ we make use of the Hermite interpolants $i_2\otimes i_2:\mathsf{H}^4(K)\to\mathbb{Q}^3(K)$ on the parallelogram and Argyris $I^5_T:\mathsf{H}^4(T)\to\mathbb{P}^5(T)$, which are such that

$$\begin{aligned}\partial^\alpha\big((i_2\otimes i_2)z\big)(N) &= \partial^\alpha z(N),\quad z\in\mathsf{H}^4(K),\quad N\text{ node of }K,\quad|\alpha|\le 2,\\ \partial^\alpha(I^5_Tz)(N) &= \partial^\alpha z(N),\quad z\in\mathsf{H}^4(T),\quad N\text{ node of }T,\quad|\alpha|\le 2.\end{aligned}$$

As for the boundary correctors, they rely on Propositions 8.6 and 8.7. Finally the inclusion (5.2) is obvious.

## 6.c Bilinearly mapped quadrilaterals

Here, the elements $K \in \mathfrak{M}^p$ are images of $\widehat{I}^2 = (0,1)^2$ under a bilinear map, i.e. $K = F_K(I^2)$, $F_K \in (\mathbb{Q}^1)^2$, or, in coordinates, $K \ni x = F_K(\widehat{x})$. The mapping $F_K$ is bijective and its Jacobian is given by

$$DF_K(\widehat{x}) = \begin{pmatrix} \dfrac{\partial x_1}{\partial \widehat{x}_1} & \dfrac{\partial x_1}{\partial \widehat{x}_2} \\ \dfrac{\partial x_2}{\partial \widehat{x}_1} & \dfrac{\partial x_2}{\partial \widehat{x}_2} \end{pmatrix}. \tag{6.25}$$

Its determinant, $J_K(\widehat{x}) = \det DF_K(\widehat{x})$ is an affine function of $\widehat{x}$.

In order to obtain a $\mathscr{C}^1$ continuous $hp$-interpolant, we need to impose a geometric condition on the mappings $F_K$. To state it, consider a patch $P = (K_+, K_-)$ of two elements sharing edge $a$ as shown in Figure 6. If $J_+$, $J_-$ denote the Jacobians of the element maps $F_{K_+}, F_{K_-}$, we assume that there is a constant $\rho \neq 0$ such that

$$J_+|_{\widehat{a}} = \rho\, J_-|_{\widehat{a}}\,. \tag{6.26}$$

This condition does not hold for arbitrary bilinear element maps. We have

**Lemma 6.2** *Consider two elements $K_+, K_-$ sharing a common edge $a$ as shown in Figure 6. If the quantities $a_\pm, b_\pm$ shown in Figure 6 satisfy $a_+/b_+ = a_-/b_-$, then condition* (6.26) *holds.*

Denoting by $x_K$ the center of $K$ and by $H_K(x)$ the homothety from Section 3.b, we have for the semi-reference element $\breve{K} = H_K^{-1}(K)$ that $F_K = H_K \circ \breve{F}_K$ where $\breve{F}_K \in \mathbb{Q}^1(\widehat{K})^2$ is independent of the diameter of element $K$, and it holds

$$J_K(\widehat{x}) = \det DF_K(\widehat{x}) = \det DH_K \det D\breve{F}_K\,.$$

Moreover, there exists $\gamma > 0$ independent of $K \in \mathfrak{M}^p$ and of $p$ such that

$$\forall \widehat{x} \in \widehat{I}^2 : \gamma^{-1} \geq \det D\breve{F}_K(\widehat{x}) \geq \gamma > 0\,. \tag{6.27}$$

We assume below that we are in a semi-reference patch and omit the "˘" from all quantities.

### *6.c.(i)* ***Primary interpolants***

The elemental approximation spaces at the level $d = 0$ are

$$V^p_{(0),K} = W^p_{(0),K} = \{v = \widehat{v} \circ F_K^{-1} : \widehat{v} \in \mathbb{Q}^p(\widehat{I}^2)\}\,, \tag{6.28}$$

and the corresponding interpolant is

$$\Pi^p_{(0),K} u := \left(\Pi^p_2 \left(u|_K \circ F_K\right)\right) \circ F_K^{-1} \tag{6.29}$$

with $\Pi^p_2$ as in Section 8.b below.

For the level $d = 1$, we define moreover the spaces

$$\begin{aligned} V^p_{(1),K} &= \{v = \widehat{v} \circ F_K^{-1} : \widehat{v} = J_K^3\, \widehat{v};\ \widehat{v} \in \mathbb{Q}^{p-4}\}\,. \\ W^p_{(1),K} &= \{v = \widehat{v} \circ F_K^{-1} : \widehat{v} = J_K^2\, \widehat{v}_2 + J_K^3\, \widehat{v}_3,\ \widehat{v}_2 \in \mathbb{Q}^{p-2},\ \widehat{v}_2 \in \mathbb{Q}^{p-4}\}. \end{aligned} \tag{6.30}$$

There holds

**Lemma 6.3** *It holds*

$$\textbf{grad}\ W^p_{(1),K} \subset W^p_{(0),K} \times W^p_{(0),K}\,.$$

PROOF. It holds, with $\widehat{\partial}_i = \frac{\partial}{\partial \widehat{x}_i},\ \partial_i = \frac{\partial}{\partial x_i}$,

$$\begin{pmatrix} \dfrac{\partial \widehat{x}_1}{\partial x_1} & \dfrac{\partial \widehat{x}_2}{\partial x_1} \\[2ex] \dfrac{\partial \widehat{x}_1}{\partial x_2} & \dfrac{\partial \widehat{x}_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} \widehat{\partial}_1 \\[2ex] \widehat{\partial}_2 \end{pmatrix} = \begin{pmatrix} \partial_1 \\[2ex] \partial_2 \end{pmatrix} = (DF_K)^{-1,\top}(\widehat{x}) \begin{pmatrix} \widehat{\partial}_1 \\[2ex] \widehat{\partial}_2 \end{pmatrix}.$$

For $v = \widehat{v} \circ F_K^{-1}$, we have $\mathbf{w} = \textbf{grad}\ v = \widehat{\mathbf{w}} \circ F_K^{-1}$ with

$$\widehat{\mathbf{w}} = (DF_K(\widehat{x}))^{-1,\top}\ \widehat{\textbf{grad}}\ \widehat{v} = \frac{1}{J_K(\widehat{x})}\ M_K(\widehat{x})\ \widehat{\textbf{grad}}\ \widehat{v}$$

where $M_K \in (\mathbb{P}^1)^4$. For $\widehat{v} = J_K^j\, \widehat{v}_j$, $\widehat{v}_j \in \mathbb{Q}^{p-j}$, we find

$$\widehat{\textbf{grad}}\ \widehat{v} = J_K^j\ \widehat{\textbf{grad}}\ \widehat{v}_j + j\ J_K^{j-1}\, (\widehat{\textbf{grad}}\ J_K)\, \widehat{v}_j$$

and hence for $j \geq 2$ that

$$\widehat{\mathbf{w}} = J_K^{j-1}\ M_K\ \widehat{\textbf{grad}}\ \widehat{v}_j + j\ J_K^{j-2}\, M_K\, (\widehat{\textbf{grad}}\ J_K)\, \widehat{v}_j\,.$$

If $\widehat{v}_j \in \mathbb{Q}^{p-j}$, this expression shows that $\widehat{\mathbf{w}} \in (\mathbb{Q}^p)^2$. ∎

We define the elemental interpolant $\Pi^p_{(1),K}$ through a modification of $\Pi^p_{(0),K}$. We set first

$$P^p_{(1),K} u := J_K^3\, \Pi^{p-4}_{(1),K}(u J_K^{-3})\,, \tag{6.31}$$

and we modify $P^p_{(1),K}$ so to satisfy nodal interpolation properties of order $2$. Note that $P^p_{(1),K}$ has the same approximation properties as $\Pi^p_{(0),K}$, i.e. (4.2) holds also for $P^p_{(1),K}$. We cocatenate $P^p_{(1),K}$ with a Hermite interpolant $I_K^5$ such that for any $u \in \mathscr{C}^2(\overline{K})$,

$$I_K^5 u \ \text{ has the form }\ J_K^3 v \ \text{ with }\ v = \widehat{v} \circ F_K^{-1} \ \text{ and }\ \widehat{v} \in \mathbb{Q}^5(\widehat{I}^2)$$

and there holds for all nodes $N$ of $K$

$$\big(\partial^\alpha I_K^5 u\big)(N) = \big(\partial^\alpha u\big)(N), \quad \forall \alpha,\ |\alpha| \leq 2\,.$$

Finally we set

$$\Pi^p_{(1),K} = P^p_{(1),K} - I_K^5 \circ P^p_{(1),K}. \tag{6.32}$$

Using the approximation properties of $P^p_{(1),K}$ and the stability of $I_K^5$, we see that $(\mathrm{I}_1)$ holds.

### *6.c.(ii) **Interface correctors***

Let $P = (K_+, K_-)$ be a patch. We denote by $J_+, J_-$ the Jacobian determinants of $F_{K_+}$, $F_{K_-}$. We are in the situation shown in Fig. 6.

The spaces $V^p_{(d),\mathrm{nod}}(P)$ are like in (6.23). Let $z \in C^\infty(\overline{U}_P)$. Since the case $d = 0$ is standard, we fix $d = 1$ and construct in two steps a $\mathscr{C}^1$-lifting of

$$V^{(1)} := \Pi^p_{(1)}(P)z.$$

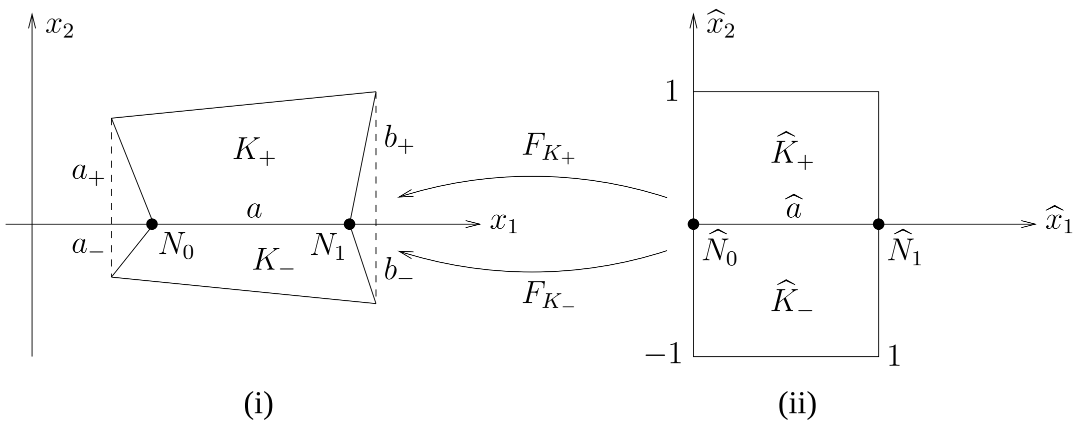


Figure 6: Patch $P = (K_+, K_-)$ of two bilinearly mapped quadrilaterals sharing an edge $a$, and reference patch $\widehat{P} = (\widehat{K}_+, \widehat{K}_-)$

*(a) Lifting of* $\left[V^{(1)}\right]_a$. Writing $V^{(1)}_\pm = V^{(1)}|_{K_\pm}$, we have

$$V^{(1)}_+ \circ F_{K_+} = J^3_+\, \widehat{V}^{(1)}_+, \quad V^{(1)}_- \circ F_{K_-} = J^3_-\, \widehat{V}^{(1)}_-\,,$$

with $\widehat{V}^{(1)}_\pm \in \mathbb{Q}^{p-4}$. Noting that $F_{K_+}|_{\widehat{a}} = F_{K_-}|_{\widehat{a}}$, we construct the lifting $R^{(1)}$ of $[z - V^{(1)}]_a$ in the reference patch $\widehat{P}$, for convenience. We have

$$\left[V^{(1)}_+ - V^{(1)}_-\right]_a \circ F_{K_\pm}|_{\widehat{a}} = \left[J^3_+\, \widehat{V}^{(1)}_+ - J^3_-\, \widehat{V}^{(1)}_-\right]_{\widehat{a}}.$$

Under assumption (6.26) then

$$\left[V^{(1)}_+ - V^{(1)}_-\right]_a \circ F_{K_+}|_{\widehat{a}} = J^3_+\left[\widehat{V}^{(1)}_+ - \rho\, \widehat{V}^{(1)}_-\right]_{\widehat{a}},$$

and, since $V^{(1)}$ belongs to $V^p_{(d),\mathrm{nod}}(P)$, the jump $[\widehat{V}^{(1)}_+ - \rho\, \widehat{V}^{(1)}_-]_{\widehat{a}}$ vanishes to order two at $\widehat{N}_0$ and $\widehat{N}_1$.

Let $\widehat{\chi}(\widehat{x}_1) \in \mathbb{P}^5(0,1)$ satisfy $\widehat{\chi}^{(j)}(0) = \delta_{0,j}$ and $\widehat{\chi}^{(j)}(1) = 0$, $j = 0, 1, 2$. Then we define

$$R^{(1)}\, V^{(1)} = \begin{cases} \left(J^3_+(\widehat{V}^{(1)}_+ - \rho\, \widehat{V}^{(1)}_-)(\widehat{x}_1, 0)\, \widehat{\chi}(\widehat{x}_2)\right) \circ F^{-1}_{K_+} & \text{in } K_+\,, \\ 0 & \text{in } K_- \end{cases} \tag{6.33}$$

and set

$$V^{(2)} := V^{(1)} + R^{(1)}\, V^{(1)}\,. \tag{6.34}$$

By construction, $V^{(2)} \in C^0(\overline{U}_P)$, and we have still

$$D^\alpha V^{(2)}(N_i) = (D^\alpha z)(N_i) \text{ for } i = 0,1,\ |\alpha| \le 2\,. \tag{6.35}$$

*(b) Lifting of* $\left[\partial_{x_2} V^{(2)}\right]_a$. By (6.27), the maps $F_{K_\pm}$ are nondegenerate in $\overline{K}_\pm$ and the directional derivatives $\partial_{x_2}, \partial_{\widehat{x}_2}$ are nontangential to edge $a$. With

$$V^{(2)}_\pm \circ F_{K_\pm} = J^3_\pm \widehat{V}^{(2)}_\pm$$

we have

$$(\partial_{x_2} V^{(2)}_+ \circ F_{K_+})|_{\widehat{a}} = J^2_+(p_{+,1}\,\partial_{\widehat{x}_1} V^{(2)}_+ + p_{+,2}\,\partial_{\widehat{x}_2} V^{(2)}_+) + J_+\, p_{+,3}\, V^{(2)}_+ \tag{6.36}$$

and

$$(\partial_{x_2} V^{(2)}_- \circ F_{K_-})|_{\widehat{a}} = J^2_-(p_{-,1}\,\partial_{\widehat{x}_1} V^{(2)}_- + p_{-,2}\,\partial_{\widehat{x}_2} V^{(2)}_-) + J_-\, p_{-,3}\, V^{(2)}_-\,. \tag{6.37}$$

In (6.36), (6.37), $p_{\pm,1}$ and $p_{\pm,2}$ are linear and $p_{\pm,3}$ are quadratic polynomials in $\widehat{x}_1$.

We construct now the lifting $R^{(2)} V^{(2)}$ such that

$$\begin{aligned} R^{(2)} V^{(2)} \in W^p_{K_+,2}\,, &\quad R^{(2)} V^{(2)} = 0 \text{ in } K_-, \\ \partial_{x_2}(R^{(2)} V^{(2)})|_a &= [\partial_{x_2} V^{(2)}_+ - \partial_{x_2} V^{(2)}_-]_a\,, \\ R^{(2)} V^{(2)}|_a &= 0 \\ D^\alpha(R^{(2)} V^{(2)})|_b &= 0 \qquad |\alpha| \le 1,\ b \subset \partial K_\pm \backslash a\,. \end{aligned} \tag{6.38}$$

This lifting will be obtained with $\widehat{V}^{(2)} \in \mathbb{Q}^{p-2}$ such that

$$R^{(2)} V^{(2)} \circ F_{K_+} = J^2_+ \widehat{V}^{(2)} \tag{6.39}$$

and such that

$$\widehat{V}^{(2)}|_{\widehat{a}} = 0 \tag{6.40}$$

$$D^\alpha \widehat{V}^{(2)}|_{\widehat{b}} = 0 \qquad |\alpha| \le 1,\ \widehat{b} \subset \partial K_+ \backslash \widehat{a}\,,$$

and (6.38) holds, i.e.

$$\partial^j_{x_2}(R^{(2)} V^{(2)})|_a = \delta_{1j}[\partial_{x_2} V^{(2)}_+ - \partial_{x_2} V^{(2)}_-]_a \quad j = 0,1. \tag{6.41}$$

Since $\widehat{V}^{(2)}|_{\widehat{a}} \equiv 0,\ \partial_{\widehat{x}_1} \widehat{V}^{(2)}|_{\widehat{a}} \equiv 0$ and it follows from (6.36) and (6.39) that

$$(\partial_{x_2} R^{(2)} V^{(2)}) \circ F_{K_+}|_{\widehat{a}} = J_+\, p_{+,2}\, \partial_{\widehat{x}_2} \widehat{V}^{(2)}\,. \tag{6.42}$$

Since $p_{+,2} = \frac{\partial x_1}{\partial \widehat{x}_1}\big|_{\widehat{x}_2 = 0} \in \mathbb{P}^1(a)$ is independent of $\widehat{x}_1$, $p_{+,2} = \text{const} \neq 0$ on $\widehat{a}$. Using assumption (6.26), we find from (6.36) and from (6.42) the following equation for $\partial_{\widehat{x}_2} \widehat{V}^{(2)}$:

$$\begin{aligned} J_+\, p_{+,2}\, \partial_{\widehat{x}_2} \widehat{V}^{(2)} &= J^2_+(p_{+,1}\,\partial_{\widehat{x}_1} \widehat{V}^{(2)}_+ + p_{+,2}\,\partial_{\widehat{x}_2} \widehat{V}^{(2)}_+) + J_+\, p_{+,3}\, \widehat{V}^{(2)}_+ \\ &\quad - \rho^2\, J^2_+(p_{-,1}\,\partial_{\widehat{x}_1} \widehat{V}^{(2)}_- + p_{-,2}\,\partial_{\widehat{x}_2} \widehat{V}^{(2)}_-) - \rho J_+ p_{-,3}\, \widehat{V}^{(2)}_-\,. \end{aligned} \tag{6.43}$$

Since $\widehat{V}^{(2)}_{\pm} \in \mathbb{Q}^{p-4}$, (6.43) reads

$$J_+\, p_{+,2}\, \partial_{\widehat{x}_2}\, \widehat{V}^{(2)} = J_+\, \widehat{g}(\widehat{x}_1),\ \widehat{g} \in \mathbb{P}^{p-2}(\widehat{a})\,, \tag{6.44}$$

and we have

$$(\partial^\alpha\, \widehat{g})(\widehat{N}_i) = 0,\ i = 0, 1,\ |\alpha| \le 1\,. \tag{6.45}$$

We set

$$\widehat{V}^{(2)} = \begin{cases} (p_{+,2})^{-1}\, \widehat{g}(\widehat{x}_1)\, \widehat{\chi}(\widehat{x}_2) & \text{in } K_+\,, \\ 0 & \text{in } \widehat{K}_- \end{cases} \tag{6.46}$$

where $\widehat{\chi}(\xi) \in \mathbb{P}^5(\widehat{a})$ satisfies $\widehat{\chi}^{(j)}(0) = \delta_{1,j}$ and $\widehat{\chi}^{(j)}(1) = 0,\ j = 0, 1, 2.$

Then $R^{(2)}\, V^{(2)}$ defined in (6.39), (6.46) satisfies (6.40), (6.41) and $\widehat{a}, \widehat{b} \subset \partial\widehat{K}_+, \partial\widehat{K}_-$, and, since also $\partial^j_{\widehat{x}_1}(J_+\, \widehat{g})(\widehat{N}_i) = 0$ for $i, j = 0, 1$, we have $(\partial^j_{\widehat{x}_1}\widehat{g})(\widehat{N}_i) = 0$ for $i, j = 0, 1$. Then it follows that

$$D^\alpha\, \widehat{V}^{(2)}(\widehat{N}) = 0 \quad |\alpha| \le 2 \text{ for all nodes of } \widehat{K}_+, \widehat{K}_-\,,$$

and $R^{(2)}\, V^{(2)}$ in (6.39), (6.46) satisfies all conditions (6.38) and also $(D^\alpha\, R^{(2)}\, V^{(2)})(\widehat{N}) = 0$ for $|\alpha| \le 2,\ \widehat{N}$ node of $K_+$. $\square$

*6.c.(iii)* ***Conclusion***

The corner and boundary correctors are constructed with the same technique as in 6.*c(i)* for the primary interpolants. All our axioms are then satisfied.

# 7 Concluding Remarks

In the present paper, we have proved exponential convergence of conforming $hp$-FE approximations for the weighted regularization of the time-harmonic Maxwell equations in polygons. Let us conclude by emphasizing some points from the technical discussion of the preceding sections.

We assumed that the exact bilinear form of the weighted regularization (1.3) for Maxwell's equations can be computed in our $hp$-FEM – a rather strong assumption since the element stiffness matrices contain the possibly non-polynomial weight function $\underline{r}^\gamma$. Constraining the $hp$-FE approximations to vanish in the terminal layer, using mesh axiom $(\mathrm{M}_3)$ and the coercivity of the bilinear form in (1.3), a Strang-type perturbation argument together with classical error estimates for Gaussian quadrature of analytic functions shows that the exponential convergence rates are preserved even in the presence of numerical integration by product Gaussian rules with a (fixed) amount of overintegration.

Our '$hp$-axiomatic approach' contains a simple construction of $hp$-interpolation operators in the terminal mesh layers which vanish identically there. This eliminates the necessity for error bounds of low order interpolants in the terminal layer by Hardy-type inequalities.

The $hp$ mesh and element classes admissible in the present convergence analysis are – even when considered for standard elliptic boundary problems – more general than those previously given (e.g in [27, 28]). In particular, the $hp$-convergence results in [27, 28] are special cases of our Axioms on meshes and local polynomial spaces. The $\mathscr{C}^1$-conforming $hp$-interpolant on bilinearly mapped quadrilateral meshes which is needed in the approximation of the potentials implies also exponential convergence of the $hp$-FEM for the biharmonic problem on mapped quadrilaterals in 2-d. The construction of $\mathscr{C}^1$-conforming $hp$-interpolants on nonaffine, bilinearly mapped geometric meshes of quadrilaterals is, to our knowledge, new and implies exponential convergence of $hp$-FEM for Kirchhoff type plate models. This generalizes what has been known for triangular meshes [29].

Moreover, the results are *not* limited to geometric meshes of type a), b) and c) – in fact, our proof technique gives exponential rates of convergence also on geometric mesh families with a mixture of any of the above element types, as e.g. triangles and bilinearly mapped quadrilaterals, of triangles and affine quadrilaterals with hanging nodes. Our concept of 'semi-reference elements' allows also to treat *curved boundaries* for domains which are parametrized by a fixed number of analytic patch maps stemming, for example, from NURBS-type CAD models of the computational domain. This is confirmed by numerical experiments in [23].

# 8 Appendix: polynomial interpolants and trace liftings

We gather in this section the technical material relating to projection operators and trace liftings in polynomial spaces necessary for the proof of the previous results. This material mainly comes from [29, 34].

## 8.a Polynomial approximation results in one dimension

Let $\widehat{I} = (-1, 1)$ and $p \geq 0$ be a polynomial degree and $\mathbb{P}^p$ the set of all polynomials of degree at most $p$ in $\widehat{I}$. We have the following basic approximation result [34], Theorem 3.3.

**Lemma 8.1** *i) Every $u \in \mathsf{L}^2(\widehat{I})$ can be written as Legendre series*

$$u(x) = \sum_{n=0}^{\infty} u_n L_n(x), \quad u_n = \frac{2n+1}{2} \int_{-1}^{1} u(x)\, L_n(x) \mathrm{d}x\,, \tag{8.1}$$

*which gives sense to the operator $\pi^p$ defined as the truncated Legendre series*

$$(\pi^p u)(x) = \sum_{n=0}^{p} u_n\, L_n(x) \in \mathbb{P}^p\,. \tag{8.2}$$

*ii) If $u \in \mathsf{H}^k(\widehat{I})$, $k \geq 1$ integer, then there holds the estimate*

$$\|u - \pi^p u\|^2_{\mathsf{L}^2(\widehat{I})} \leq \frac{(p+1-k)!}{(p+1+k)!}\, \|u^{(k)}\|^2_{\mathsf{L}^2(\widehat{I})},\ 0 \leq k \leq p+1\,. \tag{8.3}$$

Let $d \geq 0$ be an integer. We need a projection operator $\pi^p_d$ which is stable in $\mathsf{H}^{d+1}$ norm (and satisfies error estimates for this norm too) and which keeps unchanged the traces in $\pm 1$ up to the order $d$. We start by defining the restriction $\widetilde{\pi}^p_d$ of $\pi^p_d$ to $\mathsf{H}^{d+1}_0(\widehat{I})$. We recall that

$$\forall u \in \mathsf{H}^{d+1}_0(\widehat{I}) : (D^j u)(\pm 1) = 0, \quad j = 0, \ldots, d\,. \tag{8.4}$$

Since $u^{(d+1)} \in \mathsf{L}^2(\widehat{I})$ we can define $\widetilde{\pi}_d^p u$ as

$$(\widetilde{\pi}_d^p\, u)(x) := \int_{-1}^{x} \int_{-1}^{x_1} \cdots \int_{-1}^{x_d} \pi^{p-d-1}\, u^{(d+1)}(x_{d+1})\, \mathrm{d}x_{d+1}\, \mathrm{d}x_d \ldots \mathrm{d}x_1 \,. \tag{8.5}$$

It is obvious that $(D^j \widetilde{\pi}_d^p u)(-1) = 0$ for $j = 0, \ldots, d$. Integrating by parts on $\widehat{I}$ and using (8.4) we find that $u^{(d+1)}$ is orthogonal to $\mathbb{P}^d$, therefore, if $p - d - 1 \geq d$, $\pi^{p-d-1}\, u^{(d+1)}$ is also orthogonal to $\mathbb{P}^d$. Integrating by parts in (8.5) with $x = 1$, we can deduce that $(\widetilde{\pi}_d^p\, u)(1) = 0$. We prove similarly that the other derivatives in $1$ are zero:

$$\forall u \in \mathsf{H}_0^{d+1}(\widehat{I}): \quad (D^j\, \widetilde{\pi}_d^p\, u)(\pm 1) = 0, \quad j = 0, \ldots, d\,. \tag{8.6}$$

We reduce $u \in \mathsf{H}^{d+1}(\widehat{I})$ to a function $\mathsf{H}_0^{d+1}(\widehat{I})$ by means of the Hermite type interpolant $i_d$:

**Lemma 8.2** *Let $d \geq 0$ be an integer. For every $u \in \mathsf{H}^{d+1}(\widehat{I})$ there exists a unique $i_d u \in \mathbb{P}^{2d+1}$ such that*

$$(D^j i_d u)(\pm 1) = (D^j u)(\pm 1),\ j = 0, \ldots, d\,. \tag{8.7}$$

*The operator $i_d$ is stable in $\mathsf{H}^{d+1}$ norm:*

$$\|i_d u\|_{\mathsf{H}^{d+1}(\widehat{I})} \leq C_d\, \|u\|_{\mathsf{H}^{d+1}(\widehat{I})}\,. \tag{8.8}$$

This follows directly from the unisolvency of the conditions (8.7) for interpolation in $\mathbb{P}^{2d+1}$.

Let $u \in \mathsf{H}^{d+1}(\widehat{I})$. We set for $p \geq 2d+1$:

$$\pi_d^p u := i_d u + \widetilde{\pi}_p^\ell (u - i_d u)\,. \tag{8.9}$$

Let us denote $u - i_d u$ by $\widetilde{u}$ for short. Since $\widetilde{u}$ belongs to $\mathsf{H}_0^{d+1}(\widehat{I})$, $\widetilde{u} - \widetilde{\pi}_d^p\, \widetilde{u}$ also belongs to $\mathsf{H}_0^{d+1}(\widehat{I})$ and the Poincaré inequality yields

$$\|D^j(\widetilde{u} - \widetilde{\pi}_d^p\, \widetilde{u})\|_{\mathsf{L}^2(\widehat{I})} \leq C_d\, \|\widetilde{u}^{(d+1)} - (\widetilde{\pi}_d^p\, \widetilde{u})^{(d+1)}\|_{\mathsf{L}^2(\widehat{I})}, \quad j = 0, \ldots, d+1\,. \tag{8.10}$$

Thus we find that for a constant $C_d$ independent of $p$ there holds

$$\begin{aligned} \|D^j(\widetilde{u} - \widetilde{\pi}_d^p\, \widetilde{u})\|^2_{\mathsf{L}^2(\widehat{I})} &\leq C_d^2\, \|\widetilde{u}^{(d+1)} - \pi^{p-d-1}(\widetilde{u}^{(d+1)})\|^2_{\mathsf{L}^2(\widehat{I})} \\ &\leq C_d^2\, \frac{(p-d-k)!}{(p-d+k)!}\, \|\widetilde{u}^{(k+d+1)}\|^2_{\mathsf{L}^2(\widehat{I})} \end{aligned} \tag{8.11}$$

for $0 \leq k \leq p - d$, where we used (8.3) with $\widetilde{u}^{(d+1)}$ in place of $u$ and $p - d - 1$ in place of $p$. Since $u - \pi_d^p u = \widetilde{u} - \widetilde{\pi}_d^p\, \widetilde{u}$, we then obtain for $d < k \leq p - d$:

$$\begin{aligned} \|D^j(u - \pi_d^p u)\|^2_{\mathsf{L}^2(\widehat{I})} &= \|D^j\big((u - i_d u) - \widetilde{\pi}_d^p(u - i_d u)\big)\|^2_{\mathsf{L}^2(\widehat{I})} \\ &\leq C_d^2\, \frac{(p-d-k)!}{(p-d+k)!}\, \|D^{k+d+1}(u - i_d u)\|^2_{\mathsf{L}^2(\widehat{I})} \\ &\leq C_d^2\, \frac{(p-d-k)!}{(p-d+k)!}\, \|u^{(k+d+1)}\|^2_{\mathsf{L}^2(\widehat{I})} \end{aligned}$$

since $k \geq d+1$ and $D^{2d+2}\, i_d u \equiv 0$. We have shown

**Theorem 8.3** *Let $d \geq 0$. Then, for any $p$ such that $p \geq 2d+1$ there exists an interpolant $\pi_d^p$ from $\mathsf{H}^{d+1}(\widehat{I})$ into $\mathbb{P}^p$ such that*

$$D^j(\pi_d^p u)(\pm 1) = D^j u(\pm 1), \quad j = 0, \dots, d \tag{8.12}$$

*and such that there hold the error estimates any $k$ such that $d < k \leq p-d$*

$$\|u - \pi_d^p u\|^2_{\mathsf{H}^{d+1}(\widehat{I})} \leq C_d^2 \, \frac{(p-d-k)!}{(p-d+k)!} \, \|u^{(k+d+1)}\|^2_{\mathsf{L}^2(\widehat{I})} \,. \tag{8.13}$$

We finally record a stability bound for the interpolant $\pi_d^p$.

**Proposition 8.4** *For $p \geq 2d+1$ there is $C_d > 0$ independent of $p$ such that*

$$\|\pi_d^p u\|_{\mathsf{H}^{d+1}(\widehat{I})} \leq C_d \, \|u\|_{\mathsf{H}^{d+1}(\widehat{I})} \,. \tag{8.14}$$

PROOF. Thanks to (8.9) we obtain

$$\|\pi_d^p u\|_{\mathsf{H}^{d+1}(\widehat{I})} \leq 2 \|\widetilde{\pi}_d^p(u - i_d u)\|_{\mathsf{H}^{d+1}(\widehat{I})}. \tag{8.15}$$

Moreover (8.8) gives us

$$\|u - i_d u\|_{\mathsf{H}^{d+1}(\widehat{I})} \leq (1 + C_d^1) \, \|u\|_{\mathsf{H}^{d+1}(\widehat{I})} \,. \tag{8.16}$$

Since $\widetilde{\pi}_d^p(u - i_d u) \in \mathsf{H}_0^{d+1}(\widehat{I})$, we get with the help of the Poincaré inequality

$$\begin{aligned}
\|\widetilde{\pi}_d^p(u - i_d u)\|_{\mathsf{H}^{d+1}(\widehat{I})} &\leq C_d^2 \, \|D^{d+1} \, \widetilde{\pi}_d^p(u - i_d u)\|_{\mathsf{L}^2(\widehat{I})} \\
&= C_d^2 \, \|\pi^{p-d-1}\big(D^{d+1}(u - i_d u)\big)\|_{\mathsf{L}^2(\widehat{I})} \\
&\leq C_d^2 \|D^{d+1}(u - i_d u)\|_{L^2(\widehat{I})} \leq C_d^2 \, \|u - i_d u\|_{\mathsf{H}^{d+1}(\widehat{I})} \,.
\end{aligned}$$

We conclude with inequalities (8.15) and (8.16). ∎

### 8.b Polynomial approximation in two dimensions

Polynomial approximations in two dimensions will be obtained by tensor product construction: Set $\widehat{K} = \widehat{I}_1 \times \widehat{I}_2$ in what follows and denote by $\pi_{d,1}^p$, $\pi_{d,2}^p$ the interpolation operators in (8.9) applied with respect to $x_1, x_2$. Define also for $p \geq 0$

$$\mathbb{Q}^p = \operatorname{span}\{x_1^i \, x_2^j; \; 0 \leq i, j \leq p\} = \mathbb{P}^p(\widehat{I}_1) \otimes \mathbb{P}^p(\widehat{I}_2)$$

and the Sobolev spaces $H^{\underline{\ell}}(\widehat{K})$ of functions with mixed highest derivative, $\underline{\ell} = (\ell_1, \ell_2)$,

$$\mathsf{H}^{\underline{\ell}}(\widehat{K}) = \{u \in \mathsf{L}^2(\widehat{K}) : D^\alpha u \in \mathsf{L}^2(\widehat{K}^{\,2}), \; 0 \leq \alpha_i \leq \ell_i\} \,. \tag{8.17}$$

equipped with the norm $\|u\|^2_{\mathsf{H}^{\underline{\ell}}(\widehat{K})} = \sum_{0 \leq \alpha_i \leq \ell_i} \|D^\alpha u\|^2_{\mathsf{L}^2(\widehat{K})}$.

Obviously, for every integer $\ell \geq 0$, we have the embeddings $\mathsf{H}^{2\ell}(\widehat{K}) \subseteq \mathsf{H}^{\ell,\ell}(\widehat{K}) \subseteq \mathsf{H}^{\ell}(\widehat{K})$, and for $\ell \geq 1$ we have the continuity property $\mathsf{H}^{\ell,\ell}(\widehat{K}) \subset \mathscr{C}^{\ell-1}(\overline{\widehat{K}})$.

We define for $u \in \mathsf{H}^{d+1,d+1}(\widehat{K})$ and $p \geq 2d+1$

$$\Pi_d^p u := (\pi_{d,1}^p \otimes \pi_{d,2}^p)\, u \in \mathbb{Q}^p \,. \tag{8.18}$$

Then we have

**Theorem 8.5** *For any $d \geq 0$, $p \geq 2d+1$, $\Pi_d^p$ is well defined and bounded from $\mathsf{H}^{d+1,d+1}(\widehat{K})$ into $\mathbb{Q}^p$. Moreover, for $0 \leq j_1, j_2 \leq d$ holds*

$$(D^{(j_1,j_2)}\, \Pi_d^p u)(\pm 1, \pm 1) = (D^{(j_1,j_2)} u)(\pm 1, \pm 1)\,, \tag{8.19}$$

*and we have for any $k_1, k_2$ such that $d < k_1, k_2 \leq p-d$ and for any $u \in \mathsf{H}^{k_1+d+1,k_2+d+1}(\widehat{K})$ the following error estimates with a constant $C_d$ independent of $k_1, k_2$ and of $p$:*

$$\begin{aligned} \|u - \Pi_d^p\, u\|^2_{\mathsf{H}^{d+1,d+1}(\widehat{K})} \leq\ C_d \Bigg\{ & \frac{(p-d-k_1)!}{(p-d+k_1)!}\, \|\partial_1^{k_1+d+1} u\|^2_{\mathsf{H}^{0,d+1}(\widehat{K})} \\ & + \frac{(p-d-k_2)!}{(p-d+k_2)!}\, \|\partial_2^{k_2+d+1} u\|^2_{\mathsf{H}^{d+1,0}(\widehat{K})} \Bigg\}\,. \end{aligned} \tag{8.20}$$

PROOF. By (8.18), we may write for $\underline{j} = (j_1, j_2)$ such that $j_1, j_2 \leq d+1$ (with $\|\circ\|$ denoting the $\mathsf{L}^2(\widehat{K})$ norm) and using the univariate bounds (8.13) and (8.14)

$$\begin{aligned} \|D^{\underline{j}}(u - \Pi_d^p u)\|^2 &= \|\partial_1^{j_1}\partial_2^{j_2}(u - id_1 \otimes \pi_{d,2}^p\, u + id_1 \otimes \pi_{d,2}^p u - \pi_{d,1}^p \otimes \pi_{d,2}^p u)\|^2 \\ &\leq 2\big\{\|\partial_1^{j_1} u - \pi_{d,2}^p(\partial_1^{j_1} u)\|^2_{\mathsf{H}^{0,j_2}(\widehat{K})} + \|\partial_2^{j_2}\, \pi_{d,2}^p\big(\partial_1^{j_1} u - \partial_1^{j_1}(\pi_{d,1}^p u)\big)\|^2\big\} \\ &\leq 2\|(id_2 - \pi_{d,2}^p)(\partial_1^{j_1} u)\|^2_{\mathsf{H}^{0,j_2}(\widehat{K})} + 2C_d\|\partial_1^{j_1}(u - \pi_{d,1}^p u)\|^2_{\mathsf{H}^{0,d+1}(\widehat{K})} \\ &\leq C_d \left\{ \frac{(p-d-k_2)!}{(p-d+k_2)!}\, \|\partial_1^{j_1}\, \partial_2^{k_2+d+1} u\|^2 + \frac{(p-d-k_1)!}{(p-d+k_1)!}\, \|\partial_2^{d+1}\, \partial_1^{k_1+d+1} u\|^2 \right\} \end{aligned}$$

which proves (8.20). ∎

### 8.c Polynomial trace lifting in a square

We present polynomial trace liftings from [29]. The construction is based on the polynomials $\chi_{d,i} \in \mathbb{P}^{2d+1}$ introduced in (6.1).

**Proposition 8.6** *Let $S = (0,1)^2$ and $a = \{(x_1, 0) : 0 < x_1 < 1\}$. Let us fix an integer $d \geq 0$ and fix $i$, $0 \leq i \leq d$. Let for $p \geq 2d+1$ a polynomial $\varphi_i(x_1)$ be given in $\mathbb{P}^p(a)$ such that*

$$\varphi_i^{(j)} = 0 \quad \textit{on} \quad \partial a = \{0, 1\}, \quad 0 \leq j \leq d\,. \tag{8.21}$$

*Then there exists a polynomial $\Phi_i(x_1, x_2)$ of degree $p$ in $x_1$ and of degree $2d+1$ in $x_2$ such that*

$$\partial_n^i\, \Phi_i\big|_a = \varphi_i\,, \quad \partial_n^j\, \Phi_i\big|_a = 0,\ \forall j \neq i, \quad \textit{and} \quad \partial_n^j\, \Phi_i\big|_{\partial S \setminus a} = 0,\ \forall j = 0, \ldots, d. \tag{8.22}$$

*Moreover, there is $C_d > 0$ independent of $p$ such that the following estimate holds*

$$\|\Phi_i\|_{\mathsf{H}^{d+1}(S)} \leq C_d\, \|\varphi_i\|_{\mathsf{H}^{d+1}(a)}\,. \tag{8.23}$$

PROOF. We set $\Phi_i = \chi_{d,i}(x_2)\,\varphi_i(x_1)$. Then (8.22) holds and (8.23) follows from the equality $\|\partial^a\,\Phi_i\|_{\mathsf{L}^2(S)} = \|\partial_1^{\alpha_1}\varphi_i\|_{\mathsf{L}^2(I)}\|\partial_2^{\alpha_2}\,\chi_{i,d}\|_{\mathsf{L}^2(I)}$. ∎

### 8.d Polynomial Trace Lifting in a triangle

The lifting in a triangle is obtained as in [29].

**Proposition 8.7** *Let* $T = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < x_1\}$ *and let* $a$ *be its lower edge* $\{(x_1, 0) : 0 < x_1 < 1\}$. *For a fixed* $i$, $0 \le i \le d$ *let* $\varphi_i \in \mathbb{P}^p(a)$ *be given,* $p \ge 3d$, *such that*

$$\varphi_i^{(j)}(0) = 0 \ \textit{ for } \ 0 \le j \le 2d - i \tag{8.24}$$

$$\varphi_i^{(j)}(1) = 0 \ \textit{ for } \ 0 \le j \le d\,. \tag{8.25}$$

*Then there exist* $\Phi_i(x_1, x_2)$ *of degree* $p$ *in* $x_1$ *and of degree* $2d + 1$ *in* $x_2$ *such that*

$$\partial_n^i\,\Phi_i\big|_a = \varphi_i\,, \quad \partial_n^j\,\Phi_i\big|_a = 0,\ \forall j \neq i, \quad \textit{and} \quad \partial_n^j\,\Phi_i\big|_{\partial T\setminus a} = 0,\ \forall j = 0, \ldots, d, \tag{8.26}$$

*and there is* $C_d > 0$ *independent of* $p$ *such that*

$$\|\Phi_i\|_{\mathsf{H}^{d+1}(T)} \le C_d\,\|\varphi_i\|_{\mathsf{H}^{d+1}(a)}\,. \tag{8.27}$$

PROOF. Set $\Phi_i(x_1, x_2) = x_1^i\,\chi_{d,i}(\frac{x_2}{x_1})\,\varphi_i(x_1)$. By (8.24), $\varphi_i(x_i) = x_1^{2d-i+1}\psi_i(x_1)$ for some $\psi_i \in \mathbb{P}^{p-2d+i}(a)$, and therefore $\Phi_i(x_1, x_2)$ is a polynomial in $x_1$ and $x_2$. The first part of (8.26) is evident, and the second part follows from (6.1):

$$\partial_n^j\,\Phi_i\big|_a = \partial_2^j\,\Phi_i(x_1, x_2)\big|_{x_2=0} = x_1^i\,\varphi_i(x_1)x_1^{-j}\,\chi_{d,i}^{(j)}(0) = \delta_{ij}\,x_1^{i-j}\,\varphi_i(x_1)\,.$$

To show (8.27), note that for any $0 \le j \le d + 1$

$$\|\partial_2^j\,\Phi_i(x_1, x_2)\|_{\mathsf{L}^2(T)} = \|x_1^{i-j}\,\chi_i^{(j)}\left(\frac{x_2}{x_1}\right)\varphi_i(x_1)\|_{\mathsf{L}^2(T)}\,.$$

and similar expressions for any derivative $\partial^\alpha\Phi_i$. By (8.24), $\varphi_i \in \mathsf{H}_0^{d+1}(I)$, and (8.27) follows from Hardy's inequality in $\mathsf{H}_0^{d+1}(I)$. ∎

## Bibliography


[1] M. AINSWORTH, J. COYLE. Hierarchic $hp$-edge element families for Maxwell's equations on hybrid quadrilateral/triangular meshes. *Comput. Methods Appl. Mech. Engrg.* **190**(49-50) (2001) 6709–6733.

[2] M. AINSWORTH, K. PINCHEDEZ. $hp$-approximation theory for BDFM and RT finite elements on quadrilaterals. *SIAM J. Numer. Anal.* **40**(6) (2002) 2047–2068 (electronic) (2003).

[3] T. APEL, S. NICAISE. The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges. *Math. Methods Appl. Sci.* **21**(6) (1998) 519–549.

[4] I. BABUŠKA, M. R. DORR. Error estimates for the combined $h$ and $p$ versions of the finite element method. *Numer. Math.* **37**(2) (1981) 257–277.

[5] I. BABUŠKA, B. Q. GUO. Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order. *SIAM J. Math. Anal.* **19**(1) (1988) 172–203.

[6] I. BABUŠKA, B. Q. GUO. Regularity of the solution of elliptic problems with piecewise analytic data. II. The trace spaces and application to the boundary value problems with nonhomogeneous boundary conditions. *SIAM J. Math. Anal.* **20**(4) (1989) 763–781.

[7] I. BABUŠKA, B. Q. GUO. Approximation properties of the $h$-$p$ version of the finite element method. *Comput. Methods Appl. Mech. Engrg.* **133**(3-4) (1996) 319–346.

[8] I. BABUŠKA, J. E. OSBORN. Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems. *Math. Comp.* **52**(186) (1989) 275–297.

[9] I. BABUŠKA, M. SURI. The $p$- and $h$-$p$ versions of the finite element method, an overview. *Comput. Methods Appl. Mech. Engrg.* **80**(1-3) (1990) 5–26.

[10] I. BABUŠKA, M. SURI. The $p$ and $h$-$p$ versions of the finite element method, basic principles and properties. *SIAM Rev.* **36**(4) (1994) 578–632.

[11] M. BIRMAN, M. SOLOMYAK. $L^2$-theory of the Maxwell operator in arbitrary domains. *Russ. Math. Surv.* **42 (6)** (1987) 75–96.

[12] M. BIRMAN, M. SOLOMYAK. On the main singularities of the electric component of the electromagnetic field in regions with screens. *St. Petersbg. Math. J.* **5 (1)** (1993) 125–139.

[13] D. BOFFI, M. COSTABEL, M. DAUGE, L. DEMKOWICZ. Discrete compactness for the $hp$ version of rectangular edge finite elements. ICES Report 04-29, The University of Texas at Austin 2004.

[14] D. BOFFI, L. DEMKOWICZ, M. COSTABEL. Discrete compctness for $p$ and $hp$ 2d edge finite elements. *Math. Models Methods Appl. Sci.* **13**(11) (2003) 1673–1687.

[15] P. BOLLEY, J. CAMUS, M. DAUGE. Régularité Gevrey pour le problème de Dirichlet dans des domaines à singularités coniques. *Comm. Partial Differential Equations* **10 (2)** (1985) 391–432.

[16] A. BUFFA, M. COSTABEL, M. DAUGE. Algebraic convergence for anisotropic edge elements in polyhedral domains. Preprint, Pavia N. 28-PV, Istituto di Matematica Applicata e Tecnologie Informatiche CNR 2003.

[17] A. BUFFA, M. COSTABEL, M. DAUGE. Anisotropic regularity results for Laplace and Maxwell operators in a polyhedron. *C. R. Acad. Sc. Paris, Série I* **336** (2003) 565–570.

[18] M. COSTABEL. A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.* **157 (2)** (1991) 527–541.

[19] M. COSTABEL, M. DAUGE. Maxwell and Lamé eigenvalues on polyhedra. *Math. Methods Appl. Sci.* **22** (1999) 243–258.

[20] M. COSTABEL, M. DAUGE. Singularities of electromagnetic fields in polyhedral domains. *Arch. Rational Mech. Anal.* **151**(3) (2000) 221–276.

[21] M. COSTABEL, M. DAUGE. Weighted regularization of Maxwell equations in polyhedral domains. *Numer. Math.* **93 (2)** (2002) 239–277.

[22] M. COSTABEL, M. DAUGE. Computation of resonance frequencies for Maxwell equations in non-smooth domains. In *Topics in Computational Wave Propagation*, volume 31 of *Lect. Notes Comput. Sci. Eng.*, pages 125–161. Springer, Berlin 2003.

[23] M. COSTABEL, M. DAUGE, D. MARTIN, G. VIAL. Weighted regularization of Maxwell equations – computations in curvilinear polygons. In F. BREZZI, A. BUFFA, S. CORSARO, A. MURLI, editors, *Proceedings of the 4th European Conference on Numerical Mathematics and Advanced Applications, Ischia, July 2001*, pages 273–280. Springer, Milano 2002.

[24] M. COSTABEL, M. DAUGE, S. NICAISE. Singularities of Maxwell interface problems. *M2AN Math. Model. Numer. Anal.* **33**(3) (1999) 627–649.

[25] P. FRAUENFELDER. $hp$-Finite Element Methods for anisotropically, locally refined meshes in three dimensions. PhD Thesis, ETH Zürich 2004.

[26] P. FRAUENFELDER, C. LAGE. Concepts—an object-oriented software package for partial differential equations. *M2AN Math. Model. Numer. Anal.* **36**(5) (2002) 937–951.

[27] B. GUO, I. BABUŠKA. Regularity of the solutions for elliptic problems on nonsmooth domains in $\mathbf{R}^3$. I. Countably normed spaces on polyhedral domains. *Proc. Roy. Soc. Edinburgh Sect. A* **127**(1) (1997) 77–126.

[28] B. GUO, I. BABUŠKA. Regularity of the solutions for elliptic problems on nonsmooth domains in $\mathbf{R}^3$. II. Regularity in neighbourhoods of edges. *Proc. Roy. Soc. Edinburgh Sect. A* **127**(3) (1997) 517–545.

[29] B. Q. GUO. The $h$-$p$ version of the finite element method for elliptic equations of order $2m$. *Numer. Math.* **53**(1-2) (1988) 199–224.

[30] R. HIPTMAIR. Finite elements in computational electromagnetism. *Acta Numerica* (2002) 237–339.

[31] V. A. KONDRAT'EV. Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* **16** (1967) 227–313.

[32] D. MARTIN. Mélina. Online documentation:
`http://perso.univ-rennes1.fr/daniel.martin/melina`.

[33] W. RACHOWICZ, L. DEMKOWICZ. An $hp$-adaptive finite element method for electromagnetics. II. A 3D implementation. *Internat. J. Numer. Methods Engrg.* **53**(1) (2002) 147–180.

[34] C. SCHWAB. *$p$- and $hp$-finite element methods. Theory and applications in solid and fluid mechanics*. The Clarendon Press, Oxford University Press, New York 1998.

[35] B. SZABÓ, I. BABUŠKA. *Finite element analysis*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York 1991.

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