Approximation of the inf-sup constant

{Martin.Costabel, Monique.Dauge}@univ-rennes1.fr

The Problem

The **inf-sup constant of the divergence** or **L**adyzhenskaya-**Babuška**-**B**rezzi constant:

$$\beta(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \sup_{\mathbf{v}\in H^1_0(\Omega)^d} \frac{\int_\Omega \operatorname{div}\mathbf{v}\, q}{|\mathbf{v}|_1 \, \|q\|_0}$$

Ω is a bounded domain in $\mathbb{R}^d$.

Question: Does $\beta(\Omega)$ converge when

1. the domain Ω or
2. the function spaces $X = H^1_0(\Omega)^d$ (velocities) and $M = L^2_\circ(\Omega)$ (pressures)

are approximated?

Generally: Upper semi-continuity

Choose subspaces $X_N \subset X$ and $M_N \subset M$ and define the **discrete LBB constant** as

$$\beta_N = \inf_{q\in M_N} \sup_{\mathbf{v}\in X_N} \frac{\int_\Omega \operatorname{div}\mathbf{v}\, q}{|\mathbf{v}|_1 \, \|q\|_0}$$

Theorem 1.
If $(M_N)_N$ is asymptotically dense in M, then

(USC) $\quad \limsup_{N\to\infty} \beta_N \le \beta(\Omega)$

Domain upper semi-continuity

Theorem 1 can be applied to inner approximations of the domain Ω:

Corollary.
Let $\Omega_N \subset \Omega$ and define the subspaces

$$X_N = H^1_0(\Omega_N)^d \quad\text{and}\quad M_N = L^2_\circ(\Omega_N)$$

via extension by zero. If $\operatorname{meas}(\Omega\setminus\Omega_N) \to 0$, then (USC) holds in the sense that

$$\limsup_{N\to\infty} \beta(\Omega_N) \le \beta(\Omega)\,.$$

Upper semi-continuity in FEM

For approximations of the function spaces, for example via Finite Element Methods, a consequence of Theorem 1 is that

"Discrete is never better than Continuous"

Suppose that a uniform discrete LBB condition has been shown:

$$\forall N: \quad \beta_N \ge \beta_* > 0\,.$$

Then

$$\beta_* \le \beta(\Omega)$$

References

[1] C. Bernardi, M. Costabel, M. Dauge, V. Girault : *Continuity properties of the inf-sup constant for the divergence*, arXiv : 1510.03978, to appear in SIAM J. Math. Anal.

[2] M. Costabel, M. Crouzeix, M. Dauge, Y. Lafranche : *The inf-sup constant for the divergence on corner domains* Numer. Methods Partial Differential Equations **31**(2) (2015), 439–458.

Domain Convergence

Theorem 2. Let Ω_N converge to Ω in Lipschitz norm, that is: $\mathfrak{F}_N : \Omega_N \to \Omega$ is a bi-Lipschitz homeomorphism such that $\|\nabla(\mathfrak{F}_N - \mathrm{Id})\|_{L^\infty} \to 0$.

Then $\quad \lim_{N\to\infty} \beta(\Omega_N) = \beta(\Omega)$

Polygonal approximation

Corollary. Let $\Omega \subset \mathbb{R}^2$ be piecewise $\mathscr{C}^2$, and let Ω_h be polygonal approximations of side length $\le h$ and such that corners$(\Omega) \subset$ corners(Ω_h).

Then $\quad |\beta(\Omega) - \beta(\Omega_h)| \le c(\Omega)\, h.$

FEM Convergence

Theorem 3. For the **h version FEM** on regular meshes, if

$$h_X/h_M \to 0, \quad\text{then}\quad \beta_N \to \beta(\Omega).$$

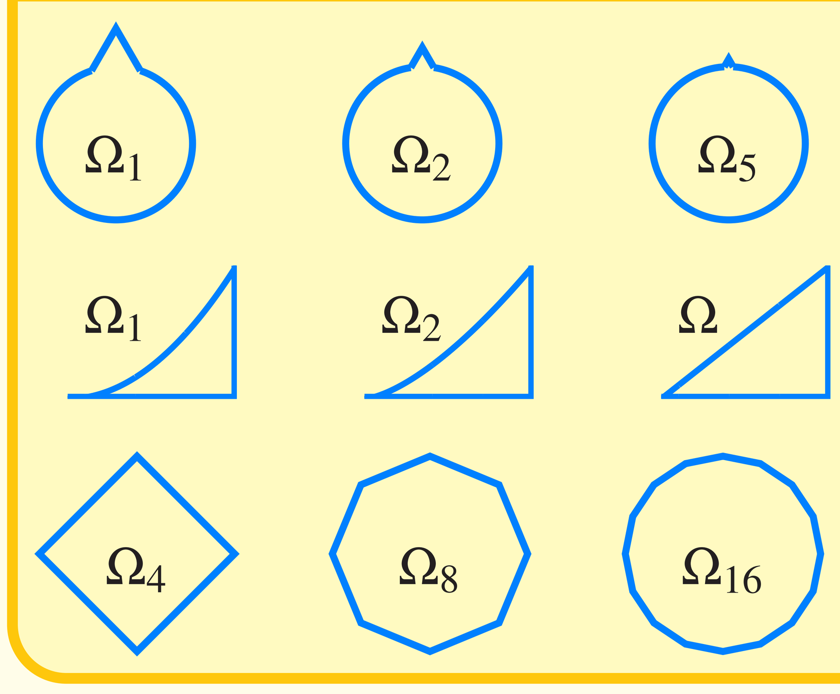
For the **p version FEM**, if

$$p_X/p_M^2 \to \infty, \quad\text{then}\quad \beta_N \to \beta(\Omega).$$

FEM Non-Convergence

Proposition. Let $\beta(\Omega) > 0$. There exists $\beta_0 > 0$ such that for any $\beta_\infty \in (0, \beta_0]$ one can find a finite element method satisfying $\lim_{N\to\infty} \beta_N = \beta_\infty$.

Examples: Domain approximation

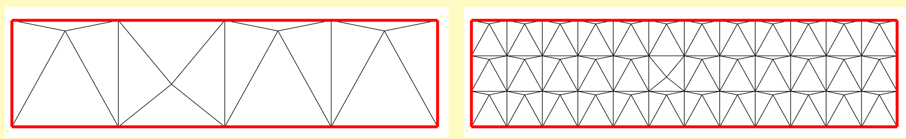


$\beta(\Omega_N) \le \beta(\text{corner}) < \sqrt{\frac{1}{2}} = \beta(\Omega)$ (disc)
$\Longrightarrow$ No convergence

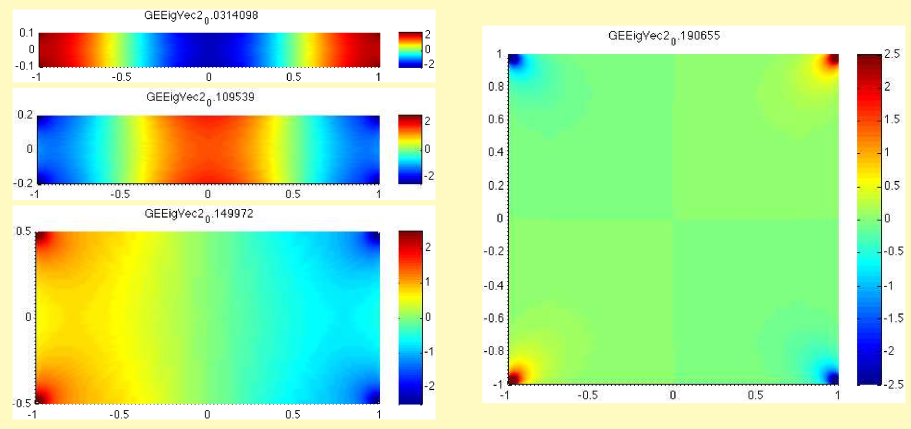
Cusps $0 < y < x^{1+1/N}$: $\beta(\Omega_N) = 0$, tend to triangle $\beta(\Omega) > 0$
$\Longrightarrow$ No convergence

Regular polygons, $0 \le \beta(\Omega) - \beta(\Omega_N) \le \frac{\pi}{2N}$: Convergence

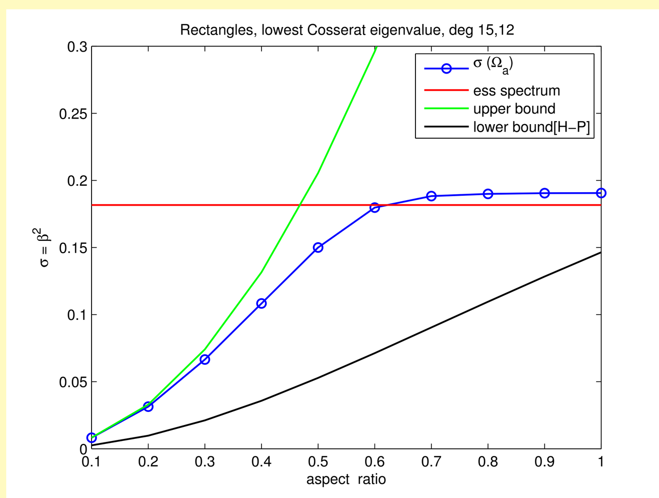
Examples: FEM approximation



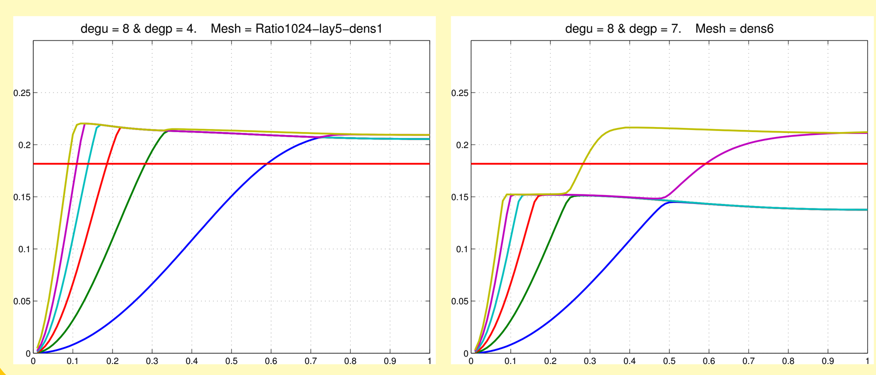
Scott-Vogelius $\mathbb{P}_4$-$\mathbb{P}_3^{\mathrm{dc}}$ elements on near-singular meshes $\Longrightarrow \lim \beta_N = \beta_\infty$ arbitrary



First Cosserat eigenfunction (pressure) on rectangles: Corner singularity depends on eigenvalue.



Computation of β^2 (lowest Cosserat eigenvalue) on rectangles with $\mathbb{Q}_{15}$-$\mathbb{Q}_{12}$ Stokes solver, refined mesh, ~ 30000 dof.
Various theoretical bounds are shown.
Red line is upper bound from continuous spectrum.
Approximation for Square is very bad!



Computation of first 4 Cosserat eigenvalues on rectangles.
Left : $p_X = 8,\ p_M = 4$
Right: $p_X = 8,\ p_M = 7$

Cosserat spectrum and corners in dimension 2: An Upper Bound

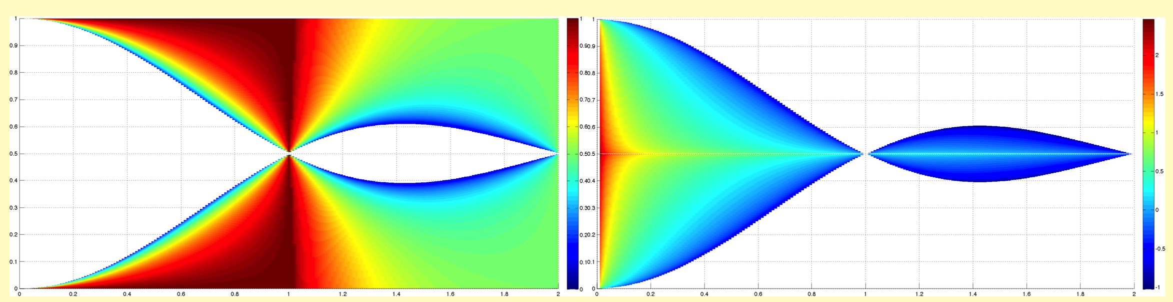
Let $\mathscr{S} = \operatorname{div}\Delta^{-1}_{\mathrm{Dir}}\nabla$ be the Schur complement operator of the Stokes system (Cosserat operator).

Then it is known that $\quad \beta(\Omega)^2 = \min \operatorname{Sp}(\mathscr{S})\,.$

If Ω has corners, then $\mathscr{S}$ has a continuous spectrum, which can be determined by Kondrat'ev's method of Mellin transformation. If the problem

$$(\sigma\Delta - \nabla\operatorname{div})\mathbf{u} = \mathbf{f}, \quad \mathbf{u} \in H^1_0(\Omega)^d$$

has corner singularities whose exponent has vanishing real part,



Real exponent as a function of ω and σ — Purely imaginary exponent as a function of ω and σ

then σ is in the continuous spectrum. For a corner of opening ω, this contributes an interval:

$$\left[\frac{1}{2} - \frac{|\sin\omega|}{2\omega},\ \frac{1}{2} + \frac{|\sin\omega|}{2\omega}\right] \subset \operatorname{Sp}(\mathscr{S}), \quad\text{hence}$$

$$\beta(\Omega)^2 \le \frac{1}{2} - \frac{|\sin\omega|}{2\omega}\,.$$