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A Direct Boundary Integral Equation Method for Transmission Problems

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A system of integral equations for the field and its normal derivative on the boundary in acoustic or potential scattering by a penetrable homogeneous object in arbitrary dimensions is presented. The system contains the operators of the single and double layer potentials, of the normal derivative of the single layer, and of the normal derivative of the double layer potential. It defines a strongly elliptic system of pseudodifferential operators. It is shown by the method of Mellin transformation that a corresponding property, namely a Gårding's inequality in the energy norm, holds also in the case of a polygonal boundary of a plane domain. This yields asymptotic quasioptimal error estimates in Sobolev spaces for the corresponding Galerkin approximation using finite elements on the boundary only.

1. INTRODUCTION

In this paper we investigate the transmission problem in $n \geqslant 2$ dimensions for the Helmholtz or Laplace equation. The transmission coefficient μ and the wave numbers k_1, k_2 are assumed to be constant complex numbers which are restricted by conditions guaranteeing the uniqueness of the solution of the transmission problem and a certain adjoint problem (Remark 4.8). The boundary Γ of the scatterer is assumed to be a smooth bounded simply connected surface in $\mathbb{R}^n$ for $n \geqslant 2$, and for $n = 2$ we also consider the case of a polygonal boundary Γ. To some extent, the results then also carry over to a curved polygon (compare [9, 12]).

We use a system of two boundary integral equations which are derived by Green's formula for the field and its normal derivative on the boundary. Thus the Cauchy data of the solution of the transmission problem are given directly by the solution of the system of integral equations, and application of the representation formula gives the solution in the whole space. In particular, in the case of a polygonal plane domain, the corner

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singularities of the solution can be explicitly calculated from the integral equations. For the exponents of the singular functions we find a transcendental equation which was derived by different methods in [48, 29].

Our system satisfies a Gårding inequality in the energy norm. This is used to show that the uniqueness assumptions on the transmission problem imply that the system of integral equations has always a unique solution. Furthermore we obtain asymptotic error estimates for general Galerkin approximation schemes on Γ (as in [46]). It is one of the main points of this paper to show by the method of Mellin transformation that this Gårding inequality remains valid also for a polygonal boundary (see Sect. 5).

Kress and Roach [33] treat the transmission problem in $\mathbb{R}^3$ by means of a different system of integral equations. They choose potentials in such a way that the most singular terms in the operators cancel and only Fredholm integral operators remain, so that the Riesz–Schauder theory applies to the system. (Compare also [31, 30, 34, 55].) This is no longer true for the case of a polygonal boundary because the operator of the double layer potential is no longer compact.

Our system resembles more an equation of the first kind with a positive definite principal part which defines in a natural way a coercive bilinear form on the energy space which is the Sobolev space $H^{1/2}(\Gamma)$ for the field and $H^{-1/2}(\Gamma)$ for its normal derivative.

We have to consider the operator of the normal derivative of the double layer potential whose kernel is hypersingular. In the case of a smooth boundary, it is a strongly elliptic pseudodifferential operator of order 1 (see [49, 40, 17, 45, 20]). The other operators of our system are the operators of the single layer potential, the double layer potential, and the normal derivative of the single layer potential, which all are pseudodifferential operators and can be handled by local Fourier transformation. In the case of a polygonal boundary in the plane, one can derive corresponding results using local Mellin transformation (see [10, 13]). In both cases one uses Sobolev spaces, and one starts with solutions in the energy space which correspond to Cauchy data of the weak solution of the transmission problem. For the equivalence of both sets of solutions, we have to assume that the homogeneous transmission problem as well as the adjoint problem obtained by interchanging the interior and exterior domains have only the trivial solution. The conditions on μ, k_1, and k_2 given in [33] are sufficient for uniqueness in the adjoint problem. Thus we need not deal with eigenvalues of interior Dirichlet problems, etc.

In order to obtain higher convergence rates for the Galerkin scheme, we study the regularity of the solutions. In the case of a polygonal boundary we derive a decomposition of the solution into corner singularities and a smooth remainder. This implies higher convergence rates for the Fix method which means that besides the standard piecewise polynomials the explicitly given corner singularities are used as test and trial functions in the Galerkin procedure (see [10, 11, 54]).

Applications of such transmission problems in acoustics and electromagnetics are described in [34, 36, 23, 37, 2, 4, 16, 39, 44, 45, 38].

The problem also appears in the scattering of time-harmonic elastic waves by a body embedded in a half space of different density, e.g., a foundation of a building. If the boundary of the body meets the free surface nonorthogonally, then the reflection method applied in [5] generates a domain with corners. In the two-dimensional case this can be treated by our boundary integral equations.

W. L. Wendland in [49, 50, 51, 52, 53] presented a list of strongly elliptic boundary integral equations, for which our system is a further example.

We want to thank Professor Wendland for many useful discussions.

2. Formulation of the Problem

Let Ω_1 denote a bounded simply connected domain in $\mathbb{R}^n$, $n \geqslant 2$, and $\Omega_2 = \mathbb{R}^n \setminus \bar{\Omega}_1$; $\Gamma = \partial\Omega_1 = \partial\Omega_2$. The interface Γ is assumed to be sufficiently smooth, for brevity C^∞, for $n \geqslant 3$ and either C^∞ or a polygon for $n = 2$. $\partial/\partial n$ denotes the derivative with respect to the normal to Γ pointing from Ω_1 to Ω_2.

We study the weak solution (u_1, u_2) of the transmission problem

$$(\Delta + k_j^2)\, u_j = 0 \qquad \text{in } \Omega_j \ (j = 1, 2) \tag{2.1}$$

$$\left.\begin{aligned} u_1 &= u_2 + v_0 \\ \mu \frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} + \psi_0 \end{aligned}\right\} \quad \text{on } \Gamma. \tag{2.2}$$

u_2 has to satisfy certain conditions at infinity:

If $k_2 \neq 0$

$$u_2(x) = \mathcal{O}(|x|^{-(n-1)/2});$$

$$\frac{\partial u_2(x)}{\partial |x|} - ik_2 u_2(x) = o(|x|^{-(n-1)/2}), \ |x| \to \infty \tag{2.3}$$

(Sommerfeld's radiation condition).

If $k_2 = 0$, for $n \geqslant 3$

$$u_2(x) = \mathcal{O}(|x|^{2-n}), \qquad |x| \to \infty; \tag{2.4}$$

for $n=2$ there are constants a and b such that

$$u_2(x)=a+\frac{b}{2\pi}\log|x|+o(1), \qquad |x|\to\infty. \tag{2.5}$$

The constants a or b may be specified (for example, $b=0$ means u bounded at infinity), and we will discuss different cases below (compare [26]).

Here k_1, k_2, and $\mu\neq 0$ are complex constants which will be subject to certain conditions below (see (4.12), (4.13), Remark 4.8). $v_0\in H^{1/2}(\Gamma)$ and $\psi_0\in H^{-1/2}(\Gamma)$ are given functions ($H^s(\Gamma)$ ($s\in\mathbb{R}$) denotes the usual Sobolev space).

In the case of scattering problems, v_0 and ψ_0 represent the boundary traces of the incident field u_0:

$$v_0=u_0|_\Gamma; \qquad \psi_0=\left.\frac{\partial u_0}{\partial n}\right|_\Gamma \text{ where } (\Delta+k_2^2)\,u_0=0 \text{ in } \Omega_1. \tag{2.6}$$

For $k_2\neq 0$, for example, $u_0(x)=e^{ik_2\xi\cdot x}$, $|\xi|=1$, represents an incident plane wave; a corresponding example for potential scattering (i.e., $k_2=0$) is given for $n=2$ by $u_0(z)=\log|z-z_0|$, $z_0\in\Omega_2$. In these cases v_0, $\psi_0\in C^\infty(\Gamma)$.

The scattered field in the exterior domain Ω_2 is u_2, and the total field u in Ω_1 is u_1, and in Ω_2 it is given by $u=u_2+u_0$. The transmission conditions (2.2) read

$$u_1=u \qquad \text{and} \qquad \mu\frac{\partial u_1}{\partial n}=\frac{\partial u}{\partial n} \text{ on } \Gamma.$$

Obviously, the transmission conditions (2.2) can easily be rewritten in the form

$$\mu_1u_1-\mu_2u_2=f; \qquad \frac{\partial u_1}{\partial n}-\frac{\partial u_2}{\partial n}=g \text{ on } \Gamma,$$

which may appear in electromagnetic scattering.

3. Cauchy Data of Weak Solutions and Calderòn Projectors

In this section, we collect standard results on Green's formula, representation formulas, and boundary integral operators in the Sobolev spaces corresponding to the weak solutions. The Sobolev spaces $H^s(\Omega_j)$ and $H^s(\Gamma)$ for smooth Γ are defined in the usual way:

$$H^s(\Omega_j)=\{u|_{\Omega_j} \mid u\in H^s(\mathbb{R}^n)\} \qquad (s\in\mathbb{R})$$
$$H^s(\Gamma)=\{u|_\Gamma \mid u\in H^{s+1/2}(\mathbb{R}^n)\} \qquad (s>0)$$
$$H^0(\Gamma)=L^2(\Gamma)$$
$$H^s(\Gamma)=(H^{-s}(\Gamma))' \qquad \text{(dual space) } (s<0).$$

If Γ is a polygon in $\mathbb{R}^2$, we will use the same definitions. Besides $H^s(\Gamma)$ we need another space $\mathscr{H}^s(\Gamma)$, defined as follows:
Let $\Gamma=\bigcup_{j=1}^J \Gamma^j$ where Γ^j are straight line segments. By z_j ($j=0,..., J$) we denote the corner points where Γ^j and Γ^{j+1} meet. (The indices will be used cyclically mod J, e.g., $z_0=z_J$.) By ω_j ($0<\omega_j<2\pi$) we denote the interior angle between Γ^j and Γ^{j+1}.

Let $s\geqslant 0$. Then $H^s(\Gamma^j)=\{u|_{\Gamma_j} \mid u\in H^s(\Gamma)\}$. We define

$$\begin{aligned}\mathscr{H}^s(\Gamma) &:= H^s(\Gamma) && \text{for } s\in[-\tfrac{1}{2},\tfrac{1}{2}) \\ &:= \{u\in L^2(\Gamma) \mid u|_{\Gamma^j}\in H^s(\Gamma^j) && (j=1,..., J)\} \text{ for } s>0.\end{aligned} \tag{3.1}$$

Now we define the spaces in which we look for the weak solutions.

$$\mathscr{L}_1 := \{u_1\in H^1(\Omega_1) \mid (\Delta+k_1^2)\,u_1=0 \text{ in } \Omega_1\}. \tag{3.2}$$

Here Δu is understood in the distributional sense. In the exterior domain we incorporate the behaviour at infinity:

$$\begin{aligned}\mathscr{L}_2 := \{u_2\in H^1_{\mathrm{loc}}(\Omega) \mid (\Delta+k_2^2)\,u_2=0 \\ \text{in } \Omega_2 \text{ and } u_2 \text{ satisfies (2.3), (2.4), (2.5)}\}.\end{aligned} \tag{3.3}$$

Note that in the case $n=2$, $k_2=0$, for any $u_2\in\mathscr{L}_2$ the constants $a, b\in\mathbb{C}$ are uniquely defined by (2.5).

The elements of $\mathscr{L}_j$ have traces on Γ in $H^{1/2}(\Gamma)$ by the usual trace lemma ([35]) for smooth Γ and by Grisvard's trace lemma ([21]) for polygonal Γ.

For the definition of the normal derivatives on Γ we use Green's formula:

LEMMA 3.1. *Let $u\in H^1_{\mathrm{loc}}(\Omega_j)$ with $\Delta u\in L^p_{\mathrm{loc}}(\Omega_j)$ ($p>1$) and $v\in H^1(\Omega_j)$ with bounded support. Then $\partial u/\partial n|_\Gamma\in H^{-1/2}(\Gamma)$ is defined by*

$$\begin{aligned}&\int_{\Omega_j} v\,\Delta u\,dx+\int_{\Omega_j}\nabla v\cdot\nabla u\,dx \\ &\qquad=(-1)^{j+1}\left\langle\frac{\partial u}{\partial n}\bigg|_\Gamma, v|_\Gamma\right\rangle_\Gamma\cdot(j=1,2).\end{aligned} \tag{3.4}$$

Here $\langle\cdot,\cdot\rangle_\Gamma$ *is the duality between* $H-1/2(\Gamma)=H^{1/2}(\Gamma))'$ *and* $H^{1/2}(\Gamma)$, *which gives* $\langle f, g\rangle_\Gamma=\int_\Gamma f(z)\, g(z)\, ds_z$ *for smooth functions* f *and* g.

The mapping $u \mapsto \partial u/\partial n|_\Gamma$ *is an extension by continuity of the corresponding natural mapping for smooth funtions.*

The proof for smooth Γ is standard, and for polynomial Γ it may be found in [42].

DEFINITION 3.2. Let $u \in H^1_{\mathrm{loc}}(\Omega_j)$ with $\Delta u \in L^p_{\mathrm{loc}}(\Omega_j)$ ($j=1, 2$). Then the "Cauchy data" $\binom{v}{\psi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ are defined to be the traces $\binom{v}{\psi}=\binom{u|_\Gamma}{\partial u/\partial n|_\Gamma}$ as defined above.

The Cauchy data of two elements of $\mathscr{L}_j$ are related:

LEMMA 3.3. *Let* $u, v \in \mathscr{L}_j$ ($j \in \{1, 2\}$), *where for* $j=2$, $n=2$, $k_2=0$ *we assume* $b=0$ *for* u *and* v, *and* $\binom{u}{\phi}$ *and* $\binom{v}{\psi}$ *be the Cauchy data of* u *and* v, *respectively. Then*

$$\langle \phi, v\rangle_\Gamma - \langle \psi, u\rangle_\Gamma = 0. \tag{3.5}$$

Proof. For bounded domains, we use the fact that Green's formula (3.4) is symmetric in u and v since

$$u\, \Delta v = -k_j^2 uv = \Delta u \cdot v.$$

This gives the assertion (3.5) for $j=1$.

For $j=2$ we choose a ball B_R with radius R and boundary S_R containing Ω_1. Then for the bounded domain $\Omega_2 \cap B_R$, (3.5) gives

$$\begin{aligned}
\langle \phi, v\rangle_\Gamma - \langle \psi, u\rangle_\Gamma &= \int_{S_R} \left(v\, \frac{\partial u}{\partial R} - u\, \frac{\partial v}{\partial R}\right) ds \\
&= ik_2 \int_{S_R} (uv - vu)\, ds + \int_{S_R} \mathcal{O}(R^{(1-n)/2})\, o(R^{(1-n)/2})\, ds \\
&= o(R^{1-n}) \cdot R^{n-1} = o(1) \qquad \text{as} \quad R \to \infty,
\end{aligned}$$

where we use the radiation condition (2.3) if $k_2 \neq 0$. The proof for $k_2=0$ and $b=0$ uses (2.4), (2.5) in the same way. ∎

For the definition of potentials, we use the following fundamental solutions which we always denote by γ_j:

$$\gamma_j(z,\zeta) := \frac{1}{2\pi}\log|z-\zeta| \qquad \text{for} \quad k_j=0,\ n=2$$

$$:= \frac{\Gamma((n-2)/2)}{4\pi^{n/2}}|z-\zeta|^{2-n} \qquad \text{for} \quad k_j=0,\ n\geqslant 3$$

$$:= -\frac{i}{4}\left(\frac{k_j}{2\pi|z-\zeta|}\right)^{(n-2)/2} H^{(1)}_{(n-2)/2}(k_j|z-\zeta|) \qquad \text{for} \quad k_j\neq 0,\ n\geqslant 2. \tag{3.6}$$

DEFINITION 3.4. Let $\phi\in C^\infty(\Gamma)$. Then

$$\left.\begin{aligned} V_{\Omega_j}\phi(z) &:= -2\int_\Gamma \gamma_j(z,\zeta)\,\phi(\zeta)\,ds_\zeta \\ K_{\Omega_j}\phi(z) &:= -2\int_\Gamma \phi(\zeta)\,\frac{\partial}{\partial n_\zeta}\gamma_j(z,\zeta)\,ds_\zeta \end{aligned}\right\} \quad (z\in\Omega_j). \tag{3.7}$$

The same definition is valid for arbitrary distributions ϕ on Γ since for $z\notin\Gamma$ the above kernels are C^∞ functions on Γ.

These potentials give the following representation formula:

LEMMA 3.5. *For $u\in\mathscr{L}_j$ with Cauchy data $\binom{v}{\psi}$ and for $z\in\Omega_j$ there holds*

$$u(z)=(-1)^j\cdot\tfrac{1}{2}\,(K_{\Omega_j}v(z)-V_{\Omega_j}\psi(z))+a, \tag{3.8}$$

where $a=0$ except for $n=2$, $k_2=0$, $j=2$. In the latter case, a is the constant appearing in (2.5).

Proof. This representation formula is well known for smooth Γ, where for exterior potential problems in two dimensions one assumes that u vanishes at infinity. The additional arguments needed for polygonal Γ for the case of bounded plane domains can be found in [10, Lemma 1.2]. For the remaining case of the exterior domain with $n=2$, where Γ may be polygonal and $k_2=0$, $a, b\neq 0$ is possible, we proceed as in the proof of Lemma 3.3. We enclose Ω_1 by a ball B_R with $R>|z|$. Then the representation formula holds for the bounded domain $\Omega_2\cap B_R$ yielding

$$u(z)=\frac{1}{2}\,(K_{\Omega_j}v(z)-V_{\Omega_j}\psi(z)) + \int_{S_R} u(\zeta)\,\frac{\partial}{\partial n_\zeta}\gamma_2(z,\zeta)\,ds_\zeta - \int_{S_R}\gamma_2(z,\zeta)\,\frac{\partial u(\zeta)}{\partial n_\zeta}\,ds_\zeta. \tag{3.9}$$

For the last two terms we use the asymptotic behaviour as $R \to \infty$:

$$u = a + \frac{b}{2\pi} \log R + \mathcal{O}(R^{-1}), \qquad \frac{\partial u}{\partial n} = \frac{b}{2\pi R} + O(R^{-2});$$

$$\gamma_2 = \frac{1}{2\pi} \log R + \mathcal{O}(R^{-1}), \qquad \frac{\partial}{\partial n} \gamma_2 = \frac{1}{2\pi R} + \mathcal{O}(R^{-2}).$$

This yields

$$\int_{S_R} u(\zeta) \frac{\partial}{\partial n_\zeta} \gamma_2(z, \zeta)\, ds_\zeta = a + \frac{b}{2\pi} \log R + o(1)$$

$$\int_{S_R} \gamma_2(z, \zeta) \frac{\partial u(\zeta)}{\partial n_\zeta}\, ds_\zeta = \frac{b}{2\pi} \log R + o(1).$$

Inserting this into (3.9) and taking the limit $R \to \infty$, we obtain (3.8). ∎

Remark 3.6. For $n = 2$, $k_2 = 0$, $j = 2$ one can get (3.8) without the constant a if one changes the fundamental solution γ_2 into

$$\tilde{\gamma}_2(z, \zeta) := \frac{1}{2\pi} \log \left| 1 - \frac{z - z_0}{\zeta - z_0} \right| \qquad \text{with some } z_0 \in \Omega_1.$$

In order to formulate the jump relations for the single and double layer potentials we define the following boundary integral operators.

DEFINITION 3.7. Let $\phi \in C^\infty(\Gamma)$. Then for $z \in \Gamma$

$$V_j\phi(z) := -2 \int_\Gamma \phi(\zeta)\, \gamma_j(z, \zeta)\, ds_\zeta;$$

$$K_j\phi(z) := -2 \int_\Gamma \phi(\zeta) \frac{\partial}{\partial n_\zeta} \gamma_j(z, \zeta)\, ds_\zeta;$$

$$K'_j\phi(z) := -2 \int_\Gamma \phi(\zeta) \frac{\partial}{\partial n_z} \gamma_j(z, \zeta)\, ds_\zeta;$$

$$D_j\phi(z) := -\frac{\partial}{\partial n_z} K_{\Omega_j}\phi(z).$$

For a distribution ϕ we define (if possible) $V_j\phi$ and $K_j\phi$ by approximating ϕ with smooth functions and $K'_j\phi$ by duality using the relation

$$\langle K'_j\phi, w\rangle_\Gamma = \langle \phi, K_j w\rangle_\Gamma \qquad \forall w \in C^\infty(\Gamma)$$

which for smooth ϕ is obviously valid.

LEMMA 3.8. *Let* $(v, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. *Then the potentials* $K_{\Omega_j} v$ *and* $V_{\Omega_j}\psi$ *belong to* $\mathcal{L}_j$, *and their Cauchy data satisfy*

$$K_{\Omega_j} v|_\Gamma = (K_j + (-1)^j)v; \qquad V_{\Omega_j}\psi|_\Gamma = V_j\psi; \tag{3.10}$$

$$\frac{\partial}{\partial n} K_{\Omega_j} v|_\Gamma = -D_j v; \qquad \frac{\partial}{\partial n} V_{\Omega_j}\psi|_\Gamma = (K_j' - (-1)^j)\psi. \tag{3.11}$$

Proof. Obviously, the potentials $V_{\Omega_j}\psi$ and $K_{\Omega_j} v$ satisfy in Ω_j the differential equation (2.1) and the conditions (2.3)–(2.5) at infinity, where for $n = 2 = j$ and $k_2 = 0$ we have $a = b = 0$ for $K_{\Omega_j} v$ and $a = 0$, $b = \langle \psi, 1 \rangle_\Gamma$ for $V_{\Omega_j}\psi$. We show now that K_{Ω_j} maps $H^{1/2}(\Gamma)$ (continuously) into $H^1_{\mathrm{loc}}(\Omega_j)$.

If Γ is smooth, we use the fact that the kernel $(\partial/\partial n_\zeta)\,\gamma_j(z, \zeta)$ has the Fourier transform $c\xi_2/(\xi_1^2 + \xi_2^2 - k_j^2)$ which is a rational symbol of order -1. It has therefore the transmission property and hence K_{Ω_j} is a potential operator in the sense of Boutet de Monvel [6], mapping $H^s(\Gamma)$ into $H^{s+1/2}_{\mathrm{loc}}(\Omega_j)$ for any $s \in \mathbb{R}$ (see Eskin's book [18, (8.18)]).

For polygonal Γ, the result $K_{\Omega_j} v \in H^1_{\mathrm{loc}}(\Omega_j)$ for $v \in H^{1/2}(\Gamma)$ will be obtained by interpolating the two cases:

$$K_{\Omega_j}\colon H^{1/2 \pm \varepsilon}(\Gamma) \to H^{1 \pm \varepsilon}_{\mathrm{loc}}(\Omega_j), \qquad \varepsilon \in (0, 1).$$

The result for the $+$ sign and $k_j = 0$ is contained in [10, Lemma 3.3], and for the $-$ sign or $k_j \neq 0$ a similar argument is valid which will be given below (Lemma 5.2). The method is based on the representation of Ω_j as an intersection of halfplanes for each of which one can apply Eskin's results.

For V_{Ω_j}, a simpler argument is valid: The transpose of the restriction mapping $H^{s+1/2}_{\mathrm{loc}}(\mathbb{R}^n) \to H^s(\Gamma)$ for $s > 0$ gives a natural embedding $\iota\colon H^{-s}(\Gamma) \to H^{-s-1/2}_{\mathrm{comp}}(\mathbb{R}^n)$ defined by $\langle \iota\psi, \phi \rangle := \langle \psi, \phi_{|\Gamma} \rangle_\Gamma$ for $\psi \in H^{-s}(\Gamma)$, $\phi \in H^{s+1/2}_{\mathrm{loc}}(\mathbb{R}^n)$. Obviously, $V_{\Omega_j}\psi$ can be viewed as the two-dimensional convolution $-2\gamma_j * (\iota\psi)$, i.e., application of the pseudodifferential operator with symbol $\sigma(V_{\Omega_j})(\xi) = c/(|\xi|^2 - k_j^2)$, $\xi \in \mathbb{R}^n \setminus \{0\}$. This operator maps $H^t_{\mathrm{comp}}(\mathbb{R}^n) \to H^{t+2}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $t \in \mathbb{R}$ ([18]). Hence $V_{\Omega_j}\colon H^{-s}(\Gamma) \to H^{-s+3/2}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $s > 0$. For $s = \frac{1}{2}$ we obtain $V_{\Omega_j}\colon H^{-1/2}(\Gamma) \to \mathcal{L}_j$. Note that this is also valid for polygonal Γ, since it depends only on the trace lemma.

The jump relations (3.10) are well known for *smooth* functions v and ψ [22]. From the trace lemma we conclude that

$$\psi \to V_{\Omega_j}\psi|_\Gamma\colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

and

$$v \to K_{\Omega_j} v|_\Gamma\colon H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

are continuous mappings. Then we can show (3.10) for $(v, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ by approximation with smooth functions, as soon as we know that

$$V_j \colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \qquad \text{and} \qquad K_j \colon H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{3.12}$$

are continuous.

Now for smooth Γ, V_j and K_j are pseudodifferential operators of order -1 [19, 18, 15, 27, 1] yielding (3.12).

For polygonal Γ, (3.12) for the case $k_j = 0$ was shown in [10]. It will be shown below (Lemma 5.2) that the differences $V_j - V$ and $K_j - K$, where V and K are defined like V_j and K_j but with $k_j = 0$, are smoothing operators mapping into $H^s(\Gamma)$ with $s > \frac{1}{2}$. Thus (3.12) is shown and this implies (3.10).

The first equality in (3.11) is just the definition of the operator D_j. In order to show the second equality, we first change the domain Ω_j in such a way that the new domain $\tilde{\Omega}_j$ has the following properties: Γ is one of the components of the boundary $\tilde{\Gamma}$ of $\tilde{\Omega}_j$, $\tilde{\Omega}_j$ is bounded, and k_j^2 is not an eigenvalue of the Dirichlet problem in $\tilde{\Omega}_j$; i.e., for any $w \in H^{1/2}(\tilde{\Gamma})$ there is exactly one solution $u \in H^1(\tilde{\Omega}_j)$ of the problem

$$(\Delta + k_j^2)\, u = 0 \qquad \text{in} \quad \tilde{\Omega}_j,\ u|_{\tilde{\Gamma}} = w.$$

Of course this is always possible by adding a suitable sphere S_R to Γ: $\tilde{\Gamma} = \Gamma \cup S_R$.

Then we extend $\psi \in H^{-1/2}(\Gamma)$ by 0 on $\tilde{\Gamma} \setminus \Gamma$ and note that the relation (3.11) on $\tilde{\Gamma}$ implies the desired relation (3.11) on Γ. Thus we may omit the distinction between Ω_j and $\tilde{\Omega}_j$.

Now we choose an arbitrary $w \in H^{1/2}(\Gamma)$ and the corresponding solution $u \in \mathscr{L}_j$ with $u|_\Gamma = w$. Then with $\phi := (\partial u / \partial n)|_\Gamma$ the representation formula (3.8) reads

$$u = (-1)^j \cdot \tfrac{1}{2}(K_{\Omega_j} w - V_{\Omega_j} \phi),$$

and with (3.10) this gives

$$(K_j - (-1)^j) w = V_j \phi.$$

Now we use Lemma 3.3 for u and $v = V_{\Omega_j} \psi \in \mathscr{L}_j$. We obtain

$$\begin{aligned}\left\langle \frac{\partial}{\partial n} V_{\Omega_j} \psi|_\Gamma, w \right\rangle_\Gamma &= \langle V_{\Omega_j} \psi|_\Gamma, \phi \rangle_\Gamma = \langle V_j \psi, \phi \rangle_\Gamma = \langle \psi, V_j \phi \rangle_\Gamma \\ &= \langle \psi, (K_j - (-1)^j) w \rangle_\Gamma = \langle (K_j' - (-1)^j) \psi, w \rangle_\Gamma, \end{aligned} \tag{3.13}$$

which shows $(\partial/\partial n)\, V_{\Omega_j}\psi|_\Gamma = (K_j' - (-1)^j)\psi$, because w was arbitrary. Here we used the symmetry of the kernel $-2\gamma_j(z, \zeta)$ of V_j, which yields $\langle V_j\psi, \phi\rangle_\Gamma = \langle \psi, V_j\phi\rangle_\Gamma$ first for smooth ϕ and ψ, and then for all $\phi, \psi \in H^{-1/2}(\Gamma)$. The last equality in (3.13) is just the definition of K_j'. ∎

LEMMA 3.9. (a) *Let $v, w \in H^{1/2}(\Gamma)$ and $\phi, \psi \in H^{-1/2}(\Gamma)$. Then there holds*

$$\langle V_j\phi, \psi\rangle_\Gamma = \langle \phi, V_j\psi\rangle_\Gamma; \qquad \langle K_j v, \psi\rangle_\Gamma = \langle v, K_j'\psi\rangle_\Gamma;$$
$$\langle D_j v, w\rangle_\Gamma = \langle v, D_j w\rangle_\Gamma. \tag{3.14}$$

(b) *The operators*

$$V_j\colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma); \qquad K_j\colon H^{1/2}(\Gamma) \to H^{1/2}(\Gamma);$$
$$K_j'\colon H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma); \qquad D_j\colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

are continuous.

Proof. With the exception of the third equality in (3.14), all statements of the lemma are clear:

The first two equalities in (3.14) as well as the continuity of V_j and K_j were shown in the preceding proof. The continuity of K_j' follows by duality, and D_j is, by definition, composed of the continuous mappings $K_{\Omega_j}\colon H^{1/2}(\Gamma) \to \mathscr{L}_j$ (Lemma 3.8) and $-(\partial/\partial n)|_\Gamma\colon \mathscr{L}_j \to H^{-1/2}(\Gamma)$ (Lemma 3.1).

Now let $v \in H^{1/2}(\Gamma)$. We may assume that the Dirichlet problem $u \in \mathscr{L}_j$, $u|_\Gamma = v$ is solvable (this can be achieved either by cutting off some ball from Ω_j as we did above, or by changing k_j and observing that D_j depends continuously on $k_j \neq 0$). For $n = 2$, $j = 2$, $k_j = 0$ we assume $b = 0$. Then the representation formula (3.8) and the jump relation (3.11) give

$$\begin{aligned} D_j v &= -\frac{\partial}{\partial n} K_{\Omega_j} v|_\Gamma = -\frac{\partial}{\partial n}\left(V_{\Omega_j}\left(\frac{\partial u}{\partial n}\bigg|_\Gamma\right) + 2(-1)^j (u - a)\right)\bigg|_\Gamma \\ &= -(K_j' - (-1)^j)\frac{\partial u}{\partial n}\bigg|_\Gamma - 2(-1)^j \frac{\partial u}{\partial n}\bigg|_\Gamma \\ &= -(K_j' - (-1)^j)\frac{\partial u}{\partial n}\bigg|_\Gamma . \end{aligned}$$

Now let $w \in H^{1/2}(\Gamma)$. Then

$$\begin{aligned} \langle D_j v, w\rangle_\Gamma &= \left\langle -(K_j' + (-1)^j)\frac{\partial u}{\partial n}\bigg|_\Gamma, w\right\rangle_\Gamma \\ &= \left\langle \frac{\partial u}{\partial n}\bigg|_\Gamma, -(K_j + (-)^j) w\right\rangle_\Gamma . \end{aligned}$$

By (3.10), this is equal to

$$\left\langle \frac{\partial u}{\partial n}\bigg|_\Gamma, -K_{\Omega_j} w \,|_\Gamma \right\rangle_\Gamma.$$

On the other hand we have $-K_{\Omega_j} w \in \mathscr{L}_j$ (with $b=0$ for $n=2=j$, $k_2=0$). Therefore we can apply Lemma 3.3 and obtain

$$\langle D_j v, w\rangle_\Gamma = \left\langle u\,|_\Gamma, -\frac{\partial}{\partial n} K_{\Omega_j} w\,|_\Gamma \right\rangle_\Gamma = \langle v, D_j w\rangle_\Gamma. \quad \blacksquare$$

Now we define the matrix of operators

$$A_j := \begin{pmatrix} -K_j & V_j \\ D_j & K_j' \end{pmatrix}. \tag{3.15}$$

From Lemma 3.9 it is clear that

$$A_j\colon \begin{matrix} H^{1/2}(\Gamma) \\ \oplus \\ H^{-1/2}(\Gamma) \end{matrix} \to \begin{matrix} H^{1/2}(\Gamma) \\ \oplus \\ H^{-1/2}(\Gamma) \end{matrix} \qquad \text{is continuous,}$$

and the relations (3.14) show that the operator A_j is skew-symmetric with respect to the bilinear form

$$B\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right) := \langle v, \psi\rangle_\Gamma - \langle w, \phi\rangle_\Gamma. \tag{3.16}$$

Note that $B(\binom{v}{-\phi}, \binom{\bar{v}}{\bar{\phi}})$ is equivalent to the norm in $H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$.

PROPOSITION 3.10. *For all v, $w \in H^{1/2}(\Gamma)$, ϕ, $\psi \in H^{-1/2}(\Gamma)$ there holds*

$$B\left(A_j\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right) = B\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, -A_j\begin{pmatrix} w \\ \psi \end{pmatrix}\right). \tag{3.17}$$

The operators A_j are the "boundary integral operators" which characterize the Cauchy data of weak solutions of the Helmholtz, resp. Laplace equation:

THEOREM 3.11. *The following statements on $\binom{v}{\psi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ are equivalent*:

(i) *$\binom{v}{\psi}$ are Cauchy data of some $u \in \mathscr{L}_j$.*

(ii) *$(1 + (-1)^j A_j)\binom{v}{\psi} = 0$.*

The right-hand side has to be replaced by $\binom{2a}{0}$ for $n = 2 = j$, $k_2 = 0$.

(iii) *There exist* $\binom{g}{h} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ *such that*

$$\begin{pmatrix} v \\ \psi \end{pmatrix} = \frac{1}{2}(1-(-1)^j A_j)\begin{pmatrix} g \\ h \end{pmatrix}\left(+\begin{pmatrix} a \\ 0 \end{pmatrix} \text{ for } n=2=j,\ k_2=0\right).$$

Proof. "(i) $\Rightarrow$ (ii)": For $u \in \mathscr{L}_j$ we use the representation formula (3.8) and express $u|_\Gamma$ and $(\partial u/\partial n)|_\Gamma$ by means of (3.10) and (3.11). This gives

$$\begin{pmatrix} v \\ \psi \end{pmatrix} = \frac{1}{2}(1-(-1)^j A_j)\begin{pmatrix} v \\ \psi \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix},$$

which is the same as (ii).

"(ii) $\Rightarrow$ (iii)": This implication is trivial. Choose $\binom{g}{h} = \binom{v}{\psi}$.

"(iii) $\Rightarrow$ (i)": Define $u \in \mathscr{L}$ by inserting $\binom{g}{h}$ into the representation formula, i.e.,

$$u = (-1)^j \tfrac{1}{2}(K_{\Omega_j} g - V_{\Omega_j} h) + a.$$

Then again by (3.10), (3.11), the Cauchy data $\binom{u|_\Gamma}{(\partial u/\partial n)|_\Gamma}$ of u satisfy

$$\begin{pmatrix} u|_\Gamma \\ (\partial u/\partial n)|_\Gamma \end{pmatrix} = \frac{1}{2}(1-(-1)^j A_j)\begin{pmatrix} g \\ h \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} v \\ \psi \end{pmatrix}.$$

Thus $\binom{v}{\psi}$ are the Cauchy data of u. ∎

COROLLARY 3.12 (compare [43]). *The operators*

$$\tfrac{1}{2}(1-(-1)^j A_j)$$

are projection operators, the so-called "Calderòn projectors" [7, 15]. *They project in* $H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ *onto the Cauchy data of the weak solutions in* $\mathscr{L}_j$. *This means in particular* $A_j^2 = 1$, *which yields the relations*

$$K_j^2 + V_j D_j = 1 = D_j V_j + K_j'^2,$$
$$-K_j V_j + V_j K_j' = 0 = -D_j K_j + K_j' D_j.$$

Proof. Let $\binom{g}{h} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ and $\binom{v}{\psi} := \frac{1}{2}(1-(-1)^j A_j)\binom{g}{h}$. Then by the theorem,

$$\begin{pmatrix} v \\ \psi \end{pmatrix} = \frac{1}{2}(1-(-1)^j A_j)\begin{pmatrix} v \\ \psi \end{pmatrix} = \left[\frac{1}{2}(1-(-1)^j A_j)\right]^2\begin{pmatrix} g \\ h \end{pmatrix}$$

which proves the projection property. Note that this is also true for $n=2=j$, $k_2=0$, because we can choose $a=0$. ∎

Remark 2.13. The Calderòn projectors for the interior and exterior problems (with the same k_j) are conjugate:

$$\tfrac{1}{2}(1+A_j)+\tfrac{1}{2}(1-A_j)=1.$$

For the case of the potential equation in two dimensions ($A := A_1 = A_2$), the projector $\frac{1}{2}(1-A)$, which is conjugate to the projector $\frac{1}{2}(1+A)$ mapping onto the Cauchy data of the interior problem, projects onto Cauchy data of harmonic functions $u_2 \in \mathscr{L}_2$ with $a=0$, $b \neq 0$ in the asymptotics (2.5). That there indeed appears the case $b \neq 0$ can be seen as follows:

For $\binom{v}{\psi}$ in the image of $\frac{1}{2}(1+A)$, i.e., for Cauchy data of harmonic functions in Ω_1, there holds

$$\int_\Gamma \psi \, ds = 0.$$

Now this is not true for every $\psi \in H^{-1/2}(\Gamma)$. Hence in the image of $\frac{1}{2}(1-A)$, there exists ψ with

$$b = \int_\Gamma \psi \, ds \neq 0.$$

For the exterior plane potential problem, especially for the Dirichlet problem, one usually looks for bounded solutions, which means $b=0$, a arbitrary. The projector onto the Cauchy data of these solutions is given by $\frac{1}{2}(1-\tilde{A})$, where $\tilde{A}$ is defined by use of the modified fundamental solution $\tilde{\gamma}_2$ of Remark 3.6. Using the formulas of Theorem 3.11 in this case, we obtain only an approximate projection in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ or a projection operator modulo constant functions.

4. The Boundary Integral Equations

Now we return to the transmission problem (2.1)–(2.5); i.e., we consider $u_j \in \mathscr{L}_j$ ($j=1, 2$) satisfying the transmission conditions (2.2), where $\binom{v_0}{\psi_0} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ are given.

By Theorem 3.11, this transmission problem is equivalent to the following relations for the Cauchy data $\binom{v_j}{\psi_j}$ of u_j:

$$(1-A_1)\begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix} = 0, \tag{4.1}$$

$$(1+A_2)\begin{pmatrix} v_2 \\ \psi_2 \end{pmatrix} = 0, \tag{4.2}$$

$$\begin{pmatrix} v_2 \\ \psi_2 \end{pmatrix} = M\begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix} - \begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix} \quad \text{with } M = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}. \tag{4.3}$$

Here we assume $a=0$ if $n=2$, $k_2=0$. We shall discuss the case $a \neq 0$ separately.

Note that (4.1)–(4.3) has the general form of any transmission problem: (4.1) and (4.2) contain the Calderòn projectors for the interior and exterior problems, respectively, and hence define the respective Cauchy data, and (4.3) contains a bounded linear operator M, the precise form of which is irrelevant for most of the following derivations.

In the case of a scattering problem, the right-hand side $\binom{\psi_0}{\psi_0}$ satisfies (2.6). By Theorem 3.11, this is equivalent to

$$(1-A_2)\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix} = 0. \tag{4.4}$$

LEMMA 4.1. *Let* $\binom{v_j}{\psi_j} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ $(j=1,2)$. *Then there exist* $u_j \in \mathscr{L}_j$ $(j=1,2)$ *satisfying the transmission conditions* (2.2) *and having Cauchy data* $\binom{v_j}{\psi_j}$ *if and only if* $\binom{v_j}{\psi_j}$ *satisfy the relations* (4.1)–(4.3). ($a=0$ *is assumed.*)

Now from the system (4.1)–(4.3) of six equations for four unknowns, we derive a system of two equations for two unknowns:
Let $\binom{v}{\psi} := \binom{v_1}{\psi_1}$. Substitution of (4.3) into (4.2) gives

$$(1+A_2)\, M\begin{pmatrix} v \\ \psi \end{pmatrix} = (1+A_2)\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}.$$

We multiply by M^{-1} from the left and subtract (4.1). We obtain the desired *boundary integral equation*

$$H\begin{pmatrix} v \\ \psi \end{pmatrix} := \frac{1}{2}(A_1 + M^{-1}A_2 M)\begin{pmatrix} v \\ \psi \end{pmatrix} = \frac{1}{2} M^{-1}(1+A_2)\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}. \tag{4.5}$$

If $\binom{v_0}{\psi_0}$ satisfy (4.4), this simplifies to

$$H\begin{pmatrix} v \\ \psi \end{pmatrix} = M^{-1}\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}. \tag{4.6}$$

LEMMA 4.2. (i) *If* $\binom{v_j}{\psi_j} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ $(j=1,2)$ *satisfy the relations* (4.1)–(4.3), *then* $\binom{v}{\psi} := \binom{v_1}{\psi_1}$ *satisfy the integral equation* (4.5).

(ii) *If* $\binom{v}{\psi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ *satisfy* (4.5), *then* $\binom{v_j}{\psi_j}$ $(j=1,2)$, *defined by*

$$\begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix} := \frac{1}{2}(1+A_1)\begin{pmatrix} v \\ \psi \end{pmatrix}; \qquad \begin{pmatrix} v_2 \\ \psi_2 \end{pmatrix} := \frac{1}{2}(1-A_2)\left[M\begin{pmatrix} v \\ \psi \end{pmatrix} - \begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}\right], \tag{4.7}$$

satisfy the relations (4.1)–(4.3).

Proof. (i) follows from the above derivation of (4.5). If $\binom{v}{\psi}$ satisfy (4.5), then the definition (4.7) together with the projection property of the Calderòn operators, i.e., $(1-A_j)(1+A_j)=0$ (Corollary 3.12), yield (4.1) and (4.2). Equation (4.7) gives

$$\begin{aligned}\begin{pmatrix}v_2\\ \psi_2\end{pmatrix}-M\begin{pmatrix}v_1\\ \psi_1\end{pmatrix}&=\frac{1}{2}\left\{(1-A_2)\,M\begin{pmatrix}v\\ \psi\end{pmatrix}-(1-A_2)\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}-M(1+A_1)\begin{pmatrix}v\\ \psi\end{pmatrix}\right\}\\ &=-\frac{1}{2}\left\{A_2M\begin{pmatrix}v\\ \psi\end{pmatrix}+MA_1\begin{pmatrix}v\\ \psi\end{pmatrix}+(1-A_2)\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}\right\}\\ &=-MH\begin{pmatrix}v\\ \psi\end{pmatrix}-\frac{1}{2}(1-A_2)\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}.\end{aligned}$$

If (4.5) holds, this can be written as

$$\begin{aligned}\begin{pmatrix}v_2\\ \psi_2\end{pmatrix}-M\begin{pmatrix}v_1\\ \psi_1\end{pmatrix}&=-\frac{1}{2}(1+A_2)\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}-\frac{1}{2}(1-A_2)\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}\\ &=-\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}.\end{aligned}$$

Hence (4.3) holds. ∎

Thus any solution $\binom{v}{\psi}$ of the boundary integral equation (4.5) generates by (4.7) a solution of the transmission problem, but $\binom{v}{\psi}$ need not be the Cauchy data of this solution. The difference $\binom{v}{\psi}-\binom{v_1}{\psi_1}$ satisfies, by (4.7), $(1+A_1)(\binom{v}{\psi}-\binom{v_1}{\psi_1})=0$; i.e., these are Cauchy data of an *exterior* problem. Therefore we define

$$\begin{pmatrix}\tilde v_1\\ \tilde\psi_1\end{pmatrix}:=M\begin{pmatrix}v\\ \psi\end{pmatrix}-\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}-\begin{pmatrix}v_2\\ \psi_2\end{pmatrix};\qquad \begin{pmatrix}\tilde v_2\\ \tilde\psi_2\end{pmatrix}=\begin{pmatrix}v\\ \psi\end{pmatrix}-\begin{pmatrix}v_1\\ \psi_1\end{pmatrix}\tag{4.8}$$

and obtain

$$\begin{pmatrix}\tilde v_1\\ \tilde\psi_1\end{pmatrix}=\frac{1}{2}(1+A_2)\left[M\begin{pmatrix}v\\ \psi\end{pmatrix}-\begin{pmatrix}v_0\\ \psi_0\end{pmatrix}\right];\qquad \begin{pmatrix}\tilde v_2\\ \tilde\psi_2\end{pmatrix}=\frac{1}{2}(1-A_1)\begin{pmatrix}v\\ \psi\end{pmatrix},$$

and with (4.3) we see $\binom{\tilde v_1}{\tilde\psi_1}=M\binom{\tilde v_2}{\tilde\psi_2}$.

Thus $\binom{\tilde v_j}{\tilde\psi_2}$ are solutions of the homogeneous "adjoint problem"

$$(1-A_2)\begin{pmatrix}\tilde v_1\\ \tilde\psi_1\end{pmatrix}=0\tag{4.9}$$

$$(1+A_1)\begin{pmatrix}\tilde v_2\\ \tilde\psi_2\end{pmatrix}=0\tag{4.10}$$

$$\begin{pmatrix}\tilde v_2\\ \tilde\psi_2\end{pmatrix}=M^{-1}\begin{pmatrix}\tilde v_1\\ \tilde\psi_1\end{pmatrix}.\tag{4.11}$$

This problem is related to the original problem (4.1)–(4.3) by interchanging k_1 with k_2 and μ with $1/\mu$ (or, equivalently, by interchanging the role of the interior and exterior domains).

Thus the equivalence follows from the

ASSUMPTION $\tilde{\mathrm{A}}$. *The homogeneous problem* (4.9)–(4.11) *has only the trivial solution* $\binom{\tilde{v}_1}{\tilde{\psi}_1}=\binom{\tilde{v}_2}{\tilde{\psi}_2}=0$.

Conversely, any nontrivial solution of (4.9)–(4.11) yields a nontrivial solution $\binom{\tilde{v}_2}{\tilde{\psi}_2}$ of (4.5) and thus gives rise to solutions of the integral equation (4.5) which do not correspond to solutions of the original transmission problem.

PROPOSITION 4.2. *The relations* (4.1)–(4.3) *and the boundary integral equation* (4.5) *for* $\binom{v}{\psi}=\binom{v_1}{\psi_1}$ *are equivalent if and only if Assumption* ($\tilde{\mathrm{A}}$) *holds.*

In order to get uniqueness of the solutions of (4.5), we have to assume uniqueness for the original transmission problem. We make the

ASSUMPTION A. *The homogeneous problem* (4.1)–(4.3) *with* $\binom{v_0}{\psi_0}=0$ *has only the trivial solution* $\binom{v_1}{\psi_1}=\binom{v_2}{\psi_2}=0$.

Then we obtain

PROPOSITION 4.3. *The integral operator*

$$H\colon H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)$$

is injective if and only if the assumptions (A) *and* ($\tilde{\mathrm{A}}$) *hold.*

In Sect. 5, Corollary 5.4, we prove an a priori estimate for the operator H. Therefore it has a closed range and finite-dimensional kernel. The following lemma shows that this also holds for the adjoint operator.

LEMMA 4.4. *The transpose of H with respect to the bilinear form B defined in* (3.16) *is the operator* $-H$: *i.e.*,

$$B\left(H\begin{pmatrix}v\\\phi\end{pmatrix},\begin{pmatrix}w\\\psi\end{pmatrix}\right)=-B\left(\begin{pmatrix}v\\\phi\end{pmatrix},H\begin{pmatrix}w\\\psi\end{pmatrix}\right)$$

for all

$$\begin{pmatrix}v\\\phi\end{pmatrix},\begin{pmatrix}w\\\psi\end{pmatrix}\in H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma).$$

Proof. By Proposition 3.10, the transpose of A_j is $-A_j$. Furthermore, by definition of M we have

$$B\left(M\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right) = \mu B\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, M^{-1}\begin{pmatrix} w \\ \psi \end{pmatrix}\right),$$

i.e., the transpose of M is μM^{-1}. Hence, the transpose of $M^{-1}A_2M$ is $-M^{-1}A_2M$, which shows the desired result by the definition of H. ∎

COROLLARY 4.5. *The operator H in $H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)$ is a Fredholm operator of index zero which is bijective if and only if Assumptions* (A) *and* ($\tilde{\mathrm{A}}$) *hold.*

We can now use our integral operator H to prove existence for the transmission problem in terms of Fredholm's alternative which was shown by Kress and Roach [33, Theorem 4.5] using a different integral operator.

COROLLARY 4.6. *In order that the transmission problem* (4.1)–(4.3) *have a solution it is necessary and sufficient that*

$$\mu\langle v_0, \psi_1\rangle_\Gamma - \langle v_1, \psi_0\rangle_\Gamma = 0$$

for all solutions $\binom{v_j}{\psi_j}$ $(j=1,2)$ *of the corresponding homogeneous problem.*

Proof. By Lemma 4.2 we know that the transmission problem has a solution to given $\binom{v_0}{\psi_0}$ if and only if the integral equation

$$H\begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

with

$$\begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{2}M^{-1}(1+A_2)\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}$$

has a solution. By Lemma 4.4 and Corollary 4.5, this is the case if and only if

$$B\left(\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} w \\ \phi \end{pmatrix}\right) = 0 \qquad \text{for all } \begin{pmatrix} w \\ \phi \end{pmatrix} \in \ker H.$$

This is equivalent to

$$0 = B\left(\frac{1}{2}M^{-1}(1+A_2)\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} w \\ \phi \end{pmatrix}\right) = \frac{1}{\mu}B\left(\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \frac{1}{2}(1-A_2)M\begin{pmatrix} w \\ \phi \end{pmatrix}\right)$$

$$= B\left(M^{-1}\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \frac{1}{2}M^{-1}(1-A_2)M\begin{pmatrix} w \\ \phi \end{pmatrix}\right).$$

$\binom{w}{\phi} \in \ker H$ means $0 = H\binom{w}{\phi} = \frac{1}{2}(1 + A_1 - M^{-1}(1-A_2)M)\binom{w}{\phi}$.

Hence we obtain the condition

$$B\left(M^{-1}\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \frac{1}{2}(1+A_1)\begin{pmatrix} w \\ \phi \end{pmatrix}\right)=0 \qquad \text{for all } \begin{pmatrix} w \\ \phi \end{pmatrix} \in \ker H.$$

Now if $\binom{w}{\phi}$ runs through $\ker H$, by Lemma 4.2, $\binom{v_1}{\psi_1}=\frac{1}{2}(1+A_1)\binom{w}{\phi}$ runs through all solutions of the homogeneous transmission problem. Thus we obtain the necessary and sufficient solvability condition

$$B\left(M^{-1}\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix}\right)=0 \qquad \text{for all such } \begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix}. \quad \blacksquare$$

Now we want to discuss some conditions sufficient for the Assumptions (A) and ($\tilde{\text{A}}$). Uniqueness proofs are given, e.g. in [33, 23, 34, 47, 56, 8]. For the following we always assume $a=0$ for the case $n=2$, $k_2=0$.

PROPOSITION 4.7. *Let $k_1, k_2 \in \mathbb{C}$ and $\mu \in \mathbb{C}\setminus\{0\}$ be such that either*

$$k_2>0 \qquad \textit{and} \qquad \operatorname{Im}\mu \leqslant 0 \qquad \textit{and} \qquad \operatorname{Im}\mu k_1^2 \geqslant 0 \tag{4.12}$$

or

$$\operatorname{Im} k_2>0 \qquad \textit{or} \qquad k_2=0 \qquad \textit{and there exist no } \alpha, \beta, \gamma, \delta>0$$
$$\textit{with } -\mu k_1^2\alpha - k_2^2\beta + \mu\gamma + \delta = 0. \tag{4.13}$$

Then the homogeneous transmission problem (4.1)–(4.3) *has only the trivial solution, i.e., Assumption* (A) *holds.*

Proof. Let B_R be a large enough ball with boundary S_R. We apply Green's formula (3.4) to Ω_1 and $\Omega_2 \cap B_R$, use the transmission condition (4.3) and eliminate the integral over Γ. We obtain

$$\int_{S_R} \frac{\partial u_2}{\partial n}\bar{u}_2\,ds = -\mu k_1^2 \int_{\Omega_1} |u_1|^2\,dx - k_2^2 \int_{\Omega_2 \cap B_R} |u_2|^2\,dx$$
$$+\mu \int_{\Omega_1} |\nabla u_1|^2\,dx + \int_{\Omega_2 \cap B_R} |\nabla u_2|^2\,dx. \tag{4.14}$$

In the case $\operatorname{Im} k_2>0$ or $k_2=0$, the left-hand side in (4.14) tends to zero for $R\to\infty$. (Here we need $a=0$ for $n=2$, $k_2=0$.) If now (4.13) is satisfied, it follows that at least one of the integrals

$$\int_{\Omega_1} |u_1|^2\,dx, \qquad \int_{\Omega_2} |u_2|^2\,dx, \qquad \int_{\Omega_1} |\nabla u_1|^2\,dx, \qquad \int_{\Omega_2} |\nabla u_2|^2\,dx$$

vanishes. In any case it follows from the transmission condition (4.3) and the conditions at infinity that $u_1 \equiv u_2 \equiv 0$.

If $k_2 > 0$, we use the radiation condition (2.3), take imaginary parts in (4.14) and obtain

$$k_2 \int_{S_R} |u_2|^2 \, ds + o(1) = -\operatorname{Im} \mu k_1^2 \int_{\Omega_1} |u_1|^2 \, dx + \operatorname{Im} \mu \int_{\Omega_1} |\nabla u_1|^2 \, dx.$$

From (4.12) it follows that the right-hand side is nonpositive, hence $\int_{S_R} |u_2|^2 \, ds = o(1)$ as $R \to \infty$, and from Rellich's theorem follows $u_2 \equiv 0$ implying $u_1 \equiv 0$.

Remark 4.8. From (4.12), (4.13) one can easily deduce conditions sufficient for both (A) and ($\tilde{\mathrm{A}}$). Note that for the problem (4.9)–(4.11) the condition $a = 0$ is always satisfied, because the potentials $V_{\Omega_2}\psi$ and $K_{\Omega_2}v$ have the asymptotic form (2.5) with $a = 0$ (in general, $b \neq 0$).

We obtain the following list of sufficient conditions for (A) and ($\tilde{\mathrm{A}}$):

(a) $k_1 = k_2 = 0$ and $\mu \in \mathbb{C} \setminus (-\infty, 0]$;

(b) $\operatorname{Im} k_1 > 0$ and $k_2 = 0$ and there are no $\alpha, \beta, \gamma > 0$ with $-\mu k_1^2 \alpha + \mu\beta + \gamma = 0$;

(c) $k_1 > 0$ and $k_2 = 0$ and $\operatorname{Im} \mu > 0$;

(d) $\operatorname{Im} k_1 > 0$ and $\operatorname{Im} k_2 > 0$ and (4.13) holds;

(e) $\operatorname{Im} k_1 > 0$ and $k_2 > 0$ and (4.12) and (4.13) hold;

(f) $k_1 > 0$ and $k_2 > 0$ and $\operatorname{Im} \mu = 0$.

The condition given by Kress & Roach [33] is (with $k_1, k_2 \neq 0$)

$$\mu \left(\frac{k_1}{k_2}\right)^2 \in \mathbb{R}$$

and

$$\mu \left(\frac{k_1}{k_2}\right)^2 \geqslant 0 \ (<0) \qquad \text{if} \quad \operatorname{Re} k_1 \operatorname{Re} k_2 \geqslant 0 \ (<0).$$

This is contained in (d)–(f) above.

Now we consider the case $n = 2$, $k_2 = 0$, where we want to drop the condition $a = 0$ which was assumed above. In particular we look for solutions bounded at infinity. This means $b = 0$ in the asymptotics (2.5) of u_2.

Let us first consider the case $k_1 = 0$. Then every constant function $u_1 \equiv u_2 \equiv a$ is a solution of the homogeneous transmission problem. Therefore for solutions of the inhomogeneous problem we may always require $a = 0$, so we have the situation studied above. On the other hand, the constant b cannot be prescribed arbitrarily: From Green's formula (3.4) follows $\langle\psi_1, 1\rangle_\Gamma = 0$, hence the transmission condition (4.3) gives

$$\Lambda\psi_2 := \langle\psi_2, 1\rangle_\Gamma = -\langle\psi_0, 1\rangle_\Gamma. \tag{4.15}$$

From the representation formula (3.8) follows $b = \langle\psi_2, 1\rangle_\Gamma = \Lambda\psi_2$, so that b is determined by the inhomogeneity ψ_0 in this case.

Now let $k_1 \neq 0$. Then we want to prescribe b and admit $a \neq 0$. Thus we have to use the general boundary relations from Theorem 3.11 instead of (4.2). We obtain

$$(1 + A_2)\begin{pmatrix} v_2 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix}; \qquad \Lambda\psi_2 = b. \tag{4.2$'$}$$

With this modification, Lemma 4.1 remains valid. This leads to the modified boundary integral equations

$$H\begin{pmatrix} v \\ \psi \end{pmatrix} = \frac{1}{2} M^{-1}\left((1 + A_2)\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix} + \begin{pmatrix} 2a \\ 0 \end{pmatrix}\right).$$

The additional relation $\Lambda\psi_2 = b$ together with the transmission condition (4.3) yields

$$\Lambda\psi = \frac{1}{\mu}(\Lambda\psi_0 + b).$$

Now we treat the constant a as an additional unknown (as in [26]) and obtain the system

$$\tilde{H}\begin{pmatrix} v \\ \psi \\ a \end{pmatrix} = F \tag{4.5$'$}$$

with

$$F = \begin{pmatrix} \frac{1}{2} M^{-1}(1 + A_2)\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix} \\ \frac{1}{\mu}(\Lambda\psi_0 + b) \end{pmatrix}$$

and the operator

$$\tilde{H} = \left(\begin{array}{c|c} H & \begin{matrix}-1\\0\end{matrix} \\ \hline 0\,\Lambda & 0 \end{array}\right) : \begin{matrix} H^{1/2}(\Gamma) \\ \oplus \\ H^{-1/2}(\Gamma) \\ \oplus \\ \mathbb{C} \end{matrix} \to \begin{matrix} H^{1/2}(\Gamma) \\ \oplus \\ H^{-1/2}(\Gamma), \\ \oplus \\ \mathbb{C} \end{matrix}$$

i.e.,

$$\tilde{H}\begin{pmatrix} v \\ \psi \\ a \end{pmatrix} = \begin{pmatrix} H\begin{pmatrix} v \\ \psi \end{pmatrix} - \begin{pmatrix} a \\ 0 \end{pmatrix} \\ \Lambda\psi \end{pmatrix}.$$

Lemma 4.2 has to be replaced by

LEMMA 4.2′. (i) *If* $\binom{v_j}{\psi_j} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ $(j=1, 2)$ *and* $a \in \mathbb{C}$ *satisfy the relations* (4.1), (4.2′), (4.3), *then* $\begin{pmatrix} v \\ \psi \\ a \end{pmatrix} = \begin{pmatrix} v_1 \\ \psi_1 \\ a \end{pmatrix}$ *satisfy the integral equation* (4.5′) *with some* $b \in \mathbb{C}$.

(i) *If* $\begin{pmatrix} v \\ \psi \\ a \end{pmatrix} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma) \oplus \mathbb{C}$ *satisfy* (4.5′), *then* $\binom{v_j}{\psi_j}$, *defined by*

$$\begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix} := \frac{1}{2}(1 + A_1)\begin{pmatrix} v \\ \psi \end{pmatrix}; \qquad \begin{pmatrix} v_2 \\ \psi_2 \end{pmatrix} := \frac{1}{2}(1 - A_2)\left[M\begin{pmatrix} v \\ \psi \end{pmatrix} - \begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}\right] + \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{4.7′}$$

satisfy the relations (4.1), (4.2′), (4.3) *with the same a and b.*

Proof. The only difference to the proof of Lemma 4.2 is that in (ii) we have to show $\Lambda\psi_2 = b$. We note that for any $\binom{w}{\phi}$, $\binom{u}{\chi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ satisfying

$$\begin{pmatrix} u \\ \chi \end{pmatrix} = \frac{1}{2}(1 + A_2)\begin{pmatrix} w \\ \phi \end{pmatrix}$$

there holds $\Lambda\chi = 0$ since $\binom{u}{\chi}$ are the Cauchy data of a solution of $\Delta u = 0$ in Ω_1. Hence with

$$\begin{pmatrix} v_2 \\ \psi_2 \end{pmatrix} = \left(1 - \frac{1}{2}(1 + A_2)\right)\left[M\begin{pmatrix} v \\ \psi \end{pmatrix} - \begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}\right] + \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad \text{from (4.7′)}$$

it follows from (4.5′)

$$\Lambda\psi_2 = \Lambda(\mu\psi) - \Lambda\psi_0 = \mu \cdot \frac{1}{\mu}(\Lambda\psi_0 + b) - \Lambda\psi_0 = b. \quad \blacksquare$$

With the definition (4.8), the "adjoint problem" (4.9)–(4.11) remains unchanged (note that $\frac{1}{2}(1+A_2)\binom{a}{0}=\binom{a}{0}$ holds).

Thus, Proposition 4.2 reads

PROPOSITION 4.2′. *The relations* (4.1), (4.2′), (4.3), *and the boundary integral equation* (4.5′) *for* $\begin{pmatrix} v \\ \psi \\ a \end{pmatrix} = \begin{pmatrix} v_1 \\ \psi_1 \\ a \end{pmatrix}$ *are equivalent if and only if Assumption* ($\tilde{\mathrm{A}}$) *holds.*

The assumption (A) is modified to

ASSUMPTION A′. *The homogeneous problem* (4.1), (4.2′), (4.3) *with* $\binom{v_0}{\psi_0}=0$ *and* $b=0$ *has only the trivial solution*

$$\begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} v_2 \\ \psi_2 \end{pmatrix} = 0, \qquad a=0.$$

PROPOSITION 4.3′. *The operator*

$$\tilde{H}\colon H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\oplus \mathbb{C} \to H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\oplus \mathbb{C}$$

is injective if and only if assumptions (A′) *and* ($\tilde{\mathrm{A}}$) *hold.*

LEMMA 4.4′. *The transpose of* $\tilde{H}$ *with respect to the bilinear form* $\mathscr{B}$ *defined by*

$$\mathscr{B}\left(\begin{pmatrix} v \\ \phi \\ a \end{pmatrix}, \begin{pmatrix} w \\ \psi \\ c \end{pmatrix}\right) := \langle v, \psi\rangle_\Gamma - \langle w, \phi\rangle_\Gamma + a\cdot c$$

is the operator $-\hat{H}$ *where*

$$\hat{H}\begin{pmatrix} v \\ \phi \\ a \end{pmatrix} = \begin{pmatrix} H\begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix} \\ \Lambda\phi \end{pmatrix}.$$

COROLLARY 4.5′. *The operator* $\tilde{H}$ *in* $H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\oplus\mathbb{C}$ *is a Fredholm operator of index zero which is bijective if and only if assumptions* (A′) *and* ($\tilde{\mathrm{A}}$) *hold.*

Let us omit the Fredholm alternative resulting from this corollary for the problem (4.1), (4.2′), (4.3).

Concerning sufficient conditions for the uniqueness assumption (A′), we notice that also for $a \neq 0$ but $b = 0$ it follows

$$\lim_{R \to \infty} \int_{S_R} \frac{\partial u_2}{\partial n} \overline{u_2}\, ds = 0,$$

which implies

$$-\mu k_1^2 \int_{\Omega_1} |u_1|^2\, dx + \mu \int_{\Omega_1} |\nabla u_1|^2\, dx + \int_{\Omega_2} |\nabla u_2|^2\, dx = 0. \qquad (4.14')$$

Now, for $k_1 \neq 0$, constant functions are no solutions in $\mathscr{L}_1$, hence, again, vanishing of one of the integrals

$$\int_{\Omega_1} |u_1|^2\, dx, \qquad \int_{\Omega_1} |\nabla u_1|^2\, dx, \qquad \int_{\Omega_2} |\nabla u_2|^2\, dx$$

implies the uniqueness assumption (A′).

Thus the conditions (b) or (c) of Remark 4.8 are also sufficient for (A′) and ($\tilde{\mathrm{A}}$).

COROLLARY 4.9. *Let $n = 2$ and one of the conditions* (b) *or* (c) *of Remark* 4.8 *be satisfied. Then for any* $\binom{v_0}{\psi_0} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ *there exists a solution* $u_j \in \mathscr{L}_j$ $(j = 1, 2)$ *of the transmission problem* (2.1), (2.2), (2.5) *which is unique if one of the following choices of the asymptotics* (2.5) *is made*:

(i) $a = 0$ *and* b *unspecified, or*

(ii) $b \in \mathbb{C}$ *prescribed arbitrarily and* a *unspecified.*

In the first case, the solution is determined from the integral equation (4.5), *in the second case from* (4.5′). *Both integral equations are uniquely solvable.*

Remark 4.10. The integral operators in (4.5) depend continuously on k_2. If k_1 and μ are such that H is invertible for $k_2 = 0$, the solution of (4.5) therefore depends continuously on k_2 (for fixed $\binom{v_0}{\psi_0}$). Thus the solution for $k_2 \neq 0$ tends for $k_2 \to 0$ to the solution of (4.5) which corresponds to case (i) above. We see that in the limiting case of vanishing wave number the solution of the two-dimensional interface problem in general has logarithmic growth at infinity.

5. GÅRDING'S INEQUALITY FOR THE BOUNDARY INTEGRAL OPERATORS

In this section we prove a Gårding inequality for the system (4.5). We define the following bilinear form:

DEFINITION 5.1. Let $\binom{v}{\phi}, \binom{w}{\psi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$.

$$a\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right) := \left\langle \mathscr{H}\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right\rangle_\Gamma$$
$$= \langle \mathscr{D}v + \mathscr{K}'\phi, w\rangle_\Gamma + \langle -\mathscr{K}v + \mathscr{V}\phi, \psi\rangle_\Gamma$$

with

$$\mathscr{H} = \begin{pmatrix} \mathscr{D} & \mathscr{K}' \\ -\mathscr{K} & \mathscr{V} \end{pmatrix}; \qquad \mathscr{D} = \frac{1}{2}\left(D_1 + \frac{1}{\mu} D_2\right); \qquad \mathscr{K}' = \frac{1}{2}(K_1' + K_2');$$
$$\mathscr{K} = \tfrac{1}{2}(K_1 + K_2); \qquad \mathscr{V} = \tfrac{1}{2}(V_1 + \mu V_2).$$

Note that

$$\mathscr{H}\begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad \text{is equivalent to} \quad H\begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} g_2 \\ g_1 \end{pmatrix}.$$

Thus $\mathscr{H}: H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \oplus H^{1/2}(\Gamma)$ is continuous, hence

$$\left| a\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right)\right| \leqslant C(\|v\|_{1/2} + \|\phi\|_{-1/2})(\|w\|_{1/2} + \|\psi\|_{-1/2}). \tag{5.1}$$

For smooth Γ, the operator $\mathscr{H}$ is a pseudodifferential operator. The operators $\mathscr{K}': H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $\mathscr{K}: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ are compact. If we define D, K, K', and V like D_j, K_j, K_j', and V_j, respectively, with $k_j = 0$, also the operators $D_j - D$: $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $V_j - V$: $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ are compact. Thus we have

$$\mathscr{H} = \mathscr{H}_1 + \mathscr{C}_1 \qquad \text{with } \mathscr{H}_1 = \begin{pmatrix} \frac{1}{2}(1 + 1/\mu)D & 0 \\ 0 & \frac{1}{2}(1+\mu)V \end{pmatrix} \text{ and } \mathscr{C}_1 \text{ compact;}$$

$$a\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right) = \frac{1}{2}\left(1 + \frac{1}{\mu}\right)\langle Dv, w\rangle_\Gamma + \frac{1}{2}(1+\mu)\langle V\phi, \psi\rangle_\Gamma$$
$$+ \mathscr{C}_1\begin{pmatrix} w \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\Big\rangle_\Gamma.$$

Now D is a pseudodifferential operator of order 1 with principal symbol $\sigma(D)(\xi)=|\xi|$ $(\xi\in\mathbb{R}^{n-1}\backslash\{0\})$, and V is a pseudodifferential operator of order -1 with principal symbol $\sigma(V)(\xi)=1/|\xi|$ $(\xi\in\mathbb{R}^{n-1}\backslash\{0\})$ (see [51, 45]).

Thus $\mathscr{H}$ is an operator elliptic in the Agmon–Douglas–Nirenberg sense with order $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and principal symbol

$$\sigma(\mathscr{H})(\xi)=\begin{pmatrix}\frac{1}{2}(1+1/\mu)\,|\xi| & 0\\ 0 & \frac{1}{2}(1+\mu)/|\xi|\end{pmatrix}.$$

For $\mu\neq -1$, $\mathscr{H}$ is strongly elliptic in the sense of [49, 52, 46], and we have the following Gårding inequality:

$$\left|a\left(\begin{pmatrix}v\\ \phi\end{pmatrix},\begin{pmatrix}\bar{v}\\ \bar{\phi}\end{pmatrix}\right)+\left\langle\mathscr{C}_2\begin{pmatrix}v\\ \phi\end{pmatrix},\begin{pmatrix}\bar{v}\\ \bar{\phi}\end{pmatrix}\right\rangle_\Gamma\right|\geqslant\gamma(\|v\|^2_{1/2}+\|\phi\|^2_{-1/2})\tag{5.2}$$

on $H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)$ with $\gamma>0$ and $\mathscr{C}_2\colon H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\to H^{-1/2}(\Gamma)\oplus H^{1/2}(\Gamma)$ compact.

Now we turn to the case where Γ is a plane polygon. We use the decomposition

$$\mathscr{H}=\mathscr{H}_0+\mathscr{C}_0\qquad\text{with } \mathscr{H}_0=\begin{pmatrix}\frac{1}{2}(1+1/\mu)D & K'\\ -K & \frac{1}{2}(1+\mu)V\end{pmatrix},\tag{5.3}$$

where the principal part operators D, K', K, and V are defined above. We shall show that $\mathscr{H}_0$ is strongly elliptic and $\mathscr{C}_0$ is compact.

We consider the operators V_{Ω_j} and K_{Ω_j} defined in (3.7). Let $V^0_{\Omega_j}$ and $K^0_{\Omega_j}$ denote the corresponding operators for $k_j=0$ and

$$V^1_{\Omega_j}=V_{\Omega_j}-V^0_{\Omega_j},\qquad K^1_{\Omega_j}=K_{\Omega_j}-K^0_{\Omega_j}.$$

The following lemma will not only imply the desired compactness of $\mathscr{C}_0$, but also fill the gap which remained in the proof of Lemma 3.8. It will also be used for the regularity results in Sect. 6. For simplicity we write $\Omega:=\Omega_j$ and assume Ω bounded, i.e., $\Omega=\Omega_1$.

LEMMA 5.2. *The following mappings are continuous*:

(i) $K^0_\Omega\colon L^2(\Gamma)\to H^{1/2}(\Omega)$;

(ii) $K^1_\Omega\colon H^s(\Gamma)\to H^{s+5/2}(\Omega)$ $(s\in(-\frac{1}{2},\frac{1}{2}))$;

(iii) $V^1_\Omega\colon H^s(\Gamma)\to H^{s+7/2}(\Omega)$ $(s\in(-\infty,\frac{1}{2}))$.

Proof. We first treat the case $s<0$ in (iii). We can apply the same argument as in the proof of Lemma 3.8:

V_Ω^1 is defined by a pseudodifferential operator with symbol

$$\sigma(V_\Omega^1)(\xi)=\frac{c_1 k_1^2}{|\xi|^2\,(|\xi|^2-k_1^2)} \qquad (\xi\in\mathbb{R}^2\setminus\{0\},\ \text{see (5.5) below}) \tag{5.4}$$

which maps $H^t_{\mathrm{comp}}(\mathbb{R}^2)$ into $H^{t+4}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $t\in\mathbb{R}$, hence

$$V_\Omega^1\colon H^s(\Gamma)\xrightarrow{\ \iota\ } H^{s-1/2}_{\mathrm{comp}}(\mathbb{R}^2)\to H^{s+7/2}_{\mathrm{loc}}(\mathbb{R}^2)\to H^{s+7/2}(\Omega) \qquad \text{for } s<0.$$

For the other statements we consider the polygonal domain Ω as an intersection of half-planes Ω^j ($j=1,\dots,J$) whose boundary lines $\tilde{\Gamma}^j$ are incident with the segments Γ^j which form the polygon Γ. We want to decompose the operators in the lemma into contributions from each Γ^j into Ω and write this as operators mapping functions on $\tilde{\Gamma}^j$ to functions in Ω^j. Thus for functions defined on Γ we have to take the restriction on Γ^j and then the extension by zero on $\tilde{\Gamma}^j\setminus\Gamma^j$. This is possible for $\phi\in H^s(\Gamma)$, $s\in(-\frac{1}{2},\frac{1}{2})$: $\phi|_{\Gamma^j}\in H^s(\Gamma^j)$;

$$\begin{aligned}\tilde{\phi}^j &:= \phi \quad &&\text{on } \Gamma^j\\ &:= 0 &&\text{on } \tilde{\Gamma}^j\setminus\Gamma^j\end{aligned} \Rightarrow \tilde{\phi}^j\in H^s_{\mathrm{comp}}(\tilde{\Gamma}^j).$$

Then we have $V_\Omega^1\phi(z)=\sum_{j=1}^{1} V_{\Omega^j}^1\tilde{\phi}^j(z)$ for $z\in\Omega$, and similarly for the other operators.

Now the operators V_{Ω^j} are potential operators in the sense of Boutet de Monvel [6] whose continuity properties can be found in Eskin's book [18, Lemmas 8.1 and 10.1].

The operators have the following symbols:

$$\begin{aligned}\sigma(V_{\Omega^j})(\xi)&=\frac{c_1}{\xi_1^2+\xi_2^2-k_1^2}, &\qquad \sigma(V_{\Omega^j}^0)(\xi)&=\frac{c_1}{\xi_1^2+\xi_2^2},\\ \sigma(K_{\Omega^j})(\xi)&=\frac{c_2\xi_2}{\xi_1^2+\xi_2^2-k_1^2}, &\qquad \sigma(K_{\Omega^j}^0)(\xi)&=\frac{c_2\xi_2}{\xi_1^2+\xi_2^2},\end{aligned} \tag{5.5}$$

where c_1, c_2 are some constants and $\xi=(\xi_1,\xi_2)$ are coordinates chosen such that ξ_1 and ξ_2 are the dual variables to the tangential and the normal variables of $\tilde{\Gamma}_j$, respectively. From (5.5) follows (5.4) and

$$\sigma(K_{\Omega^j}^1)(\xi)=\frac{c_2 k_1^2\xi_2}{|\xi|^2\,(|\xi|^2-k_1^2)}.$$

Thus $K^0_{\Omega^j}$, $K^1_{\Omega^j}$, and $V^1_{\Omega^j}$ have rational symbols of orders -1, -3, and -4, respectively. Now Eskin's results imply for an operator W with a rational symbol of order $-\alpha$ that it maps

$$W\colon H^s_{\mathrm{comp}}(\tilde{\Gamma}^j) \to H^{s+\alpha-1/2}_{\mathrm{loc}}(\Omega^j) \qquad \text{for all } s \in \mathbb{R}. \tag{5.6}$$

Note that by the above argument (see Lemma 3.8) we also have

$$W\colon H^s(\Gamma) \to H^{s+\alpha-1/2}_{\mathrm{loc}}(\mathbb{R}^2) \qquad \text{for } s<0.$$

Now (5.6) gives immediately

$$K^0_{\Omega^j}\colon H^s_{\mathrm{comp}}(\tilde{\Gamma}^j) \ni \tilde{\phi}^j \to K^0_{\Omega^j}\tilde{\phi}^j \in H^{s+1/2}_{\mathrm{loc}}(\Omega^j),$$

hence

$$K^0_{\Omega}\colon H^s(\Gamma) \to H^{s+1/2}(\Omega) \qquad \text{for} \quad s \in (-\tfrac{1}{2}, \tfrac{1}{2}),$$

hence (i). Similarly follows

$$K^1_{\Omega}\colon H^s(\Gamma) \to H^{s+5/2}(\Omega) \qquad \text{and} \qquad V^1_{\Omega}\colon H^s(\Omega) \to H^{s+7/2}(\Omega)$$
$$\text{for} \quad s \in (-\tfrac{1}{2}, \tfrac{1}{2}). \quad \blacksquare$$

Taking traces and noting that

$$D_j = -\frac{\partial}{\partial n}\bigg|_{\Gamma} K_{\Omega_j}, \qquad K'_j = \frac{\partial}{\partial n}\bigg|_{\Gamma} V_{\Omega_j},$$

we find the mapping properties of the operator

$$\mathscr{C}_0 =: \begin{pmatrix} \mathscr{D}^1 & \mathscr{K}'^1 \\ -\mathscr{K}^1 & \mathscr{V}^1 \end{pmatrix} \colon \begin{matrix} H^{1/2}(\Gamma) \\ \oplus \\ H^{-1/2}(\Gamma) \end{matrix} \to \begin{matrix} \mathscr{H}^{3/2-\varepsilon}(\Gamma) \\ \oplus \\ H^{5/2-\varepsilon}(\Gamma) \end{matrix} \qquad \text{for any } \varepsilon > 0. \tag{5.7}$$

For the definition of $\mathscr{H}^s$, see (3.1). Here $\mathscr{D}^1 := \frac{1}{2}(D_1 - D + (1/\mu)(D_2 - D))$ etc., hence we find as a corollary to Lemma 5.2: For any $s \in (-\frac{1}{2}, \frac{1}{2})$,

$$\begin{aligned} &\mathscr{D}^1 \colon H^s(\Gamma) \to \mathscr{H}^{s+1}(\Gamma); \\ &\mathscr{K}^1 \colon H^s(\Gamma) \to H^{s+2}(\Gamma); \\ &\mathscr{V}^1 \colon H^s(\Gamma) \to H^{s+3}(\Gamma); \\ &\mathscr{K}'^1 \colon H^s(\Gamma) \to \mathscr{H}^{s+2}(\Gamma). \end{aligned} \tag{5.8}$$

Clearly, $\mathscr{C}_0\colon H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \oplus H^{1/2}(\Gamma)$ is compact.

Now we prove Gårding's inequality for $\mathscr{H}_0$ (see (5.3)). If we define

$$a_0\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix}\right) := \left\langle \mathscr{H}_0 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix} \right\rangle_{\Gamma},$$

then we have

$$a_0\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\phi} \end{pmatrix}\right) = \left\langle \frac{1}{2}\left(1+\frac{1}{\mu}\right) Dv + K'\phi, \bar{v} \right\rangle_\Gamma + \left\langle -Kv + \frac{1}{2}(1+\mu) V\phi, \bar{\phi} \right\rangle_\Gamma$$

$$= \frac{1}{2}\left(1+\frac{1}{\mu}\right) \langle Dv, \bar{v}\rangle_\Gamma + \langle K'\phi, \bar{v}\rangle_\Gamma$$

$$- \langle Kv, \bar{\phi}\rangle_\Gamma + \frac{1}{2}(1+\mu)\langle V\phi, \bar{\phi}\rangle_\Gamma$$

$$= \frac{1}{2}\left(1+\frac{1}{\mu}\right) \langle Dv, \bar{v}\rangle_\Gamma + \frac{1}{2}(1+\mu)\langle V\phi, \bar{\phi}\rangle_\Gamma + 2i \operatorname{Im}\langle \phi, K\bar{v}\rangle_\Gamma.$$

Here we use the fact that the kernel of K is real Hence

$$\operatorname{Re} a_0\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\phi} \end{pmatrix}\right) = \operatorname{Re}\left\{\frac{1}{2}\left(1+\frac{1}{\mu}\right) \langle Dv, \bar{v}\rangle_\Gamma + \frac{1}{2}(1+\mu)\langle V\phi, \bar{\phi}\rangle_\Gamma\right\}.$$

Now we know that for D [13] and V [10] there hold Gårding's inequalities: There are compact operators $C_1: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $C_2: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and constants $\gamma_1, \gamma_2 > 0$ such that

$$\langle (D + C_1)\, v, \bar{v}\rangle_\Gamma \geqslant \gamma_1 \, \|v\|_{1/2}^2$$
$$\langle (V + C_2)\, \phi, \bar{\phi}\rangle_\Gamma \geqslant \gamma_2 \, \|\phi\|_{-1/2}^2$$

for any $v \in H^{1/2}(\Gamma)$, $\phi \in H^{-1/2}(\Gamma)$. Furthermore, $\langle Dv, \bar{v}\rangle$ and $\langle V\phi, \bar{\phi}\rangle$ are real because of (3.14). Hence we obtain

$$\operatorname{Re} a_0\left(\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\phi} \end{pmatrix}\right)$$
$$\geqslant \gamma_1 \operatorname{Re} \frac{1}{2}\left(1+\frac{1}{\mu}\right) \|v\|_{1/2}^2 + \gamma_2 \operatorname{Re} \frac{1}{2}(1+\mu)\, \|\phi\|_{-1/2}^2$$
$$- \operatorname{Re}\left\langle \begin{pmatrix} \frac{1}{2}(1+1/\mu)\, C_1 & 0 \\ 0 & \frac{1}{2}(1+\mu)\, C_2 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\phi} \end{pmatrix} \right\rangle_\Gamma . \tag{5.9}$$

If we combine (5.9) with (5.3) and (5.2), we obtain

THEOREM 5.3. *There exists a compact operator* $\mathscr{C}: H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \oplus H^{1/2}(\Gamma)$ *and a constant* $\gamma > 0$ *such that*

(i) *for* $\mu \neq -1$ *and smooth* Γ *there holds*

$$\left|\left\langle (\mathscr{H}+\mathscr{C})\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\phi} \end{pmatrix}\right\rangle_\Gamma\right| \geqslant \gamma(\|v\|_{1/2}^2 + \|\phi\|_{-1/2}^2);$$

(ii) *for* $\operatorname{Re}(1+1/\mu)>0$ *and* $\operatorname{Re}(1+\mu)>0$ *there holds*

$$\operatorname{Re}\left\langle (\mathscr{H}+\mathscr{C})\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\phi} \end{pmatrix}\right\rangle_\Gamma \geqslant \gamma(\|v\|_{1/2}^2 + \|\phi\|_{-1/2}^2)$$

for all $v \in H^{1/2}(\Gamma)$, $\phi \in H^{-1/2}(\Gamma)$.

In (ii), Γ *may be smooth for* $n \geqslant 2$ *or polygonal for* $n=2$.

As a corollary, we obtain an a priori estimate for H, which was used in the proof of Lemma 4.4.

COROLLARY 5.4. *For any* $\binom{v}{\phi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ *there holds*

$$(\|v\|_{1/2}^2 + \|\phi\|_{-1/2}^2)^2 \leqslant \frac{1}{\gamma}\left(\left\|H\begin{pmatrix} v \\ \phi \end{pmatrix}\right\|_{H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)} + \left\|\mathscr{C}\begin{pmatrix} v \\ \phi \end{pmatrix}\right\|_{H^{-1/2}(\Gamma)\oplus H^{1/2}(\Gamma)}\right).$$

with γ *and* $\mathscr{C}$ *as in Theorem* 5.3.

There is a different method for proving such coerciveness results which does not use the specific form of the boundary integral operator H but instead uses a bilinear form related to the "energy" of the scattering problem. This method works also for other integral equations (see [2, 19, 14]), but it does not provide the full statement of Theorem 5.3 for *all* $\binom{v}{\phi} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

For the following we assume $\mu > 0$.

Let $\binom{v_j}{\psi_j}$ be the Cauchy data of $u_j \in \mathscr{L}_j$ $(j=1,2)$ where u_1, u_2 are solutions of the transmission problem (2.1), (2.2) with $\binom{v_0}{\psi_0}$ being Cauchy data of $u_0 \in H^1(\Omega_1)$ with $(\Delta + k_2^2)\,u_0 = 0$, i.e., $\binom{v_0}{\psi_0}$ satisfy (4.4). We define the sesquilinear form

$$b\left(\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix}\right) := \langle v_0, \overline{\psi_1}\rangle_\Gamma + \frac{1}{\mu}\langle v_1, \overline{\psi_0}\rangle_\Gamma. \tag{5.10}$$

Then the transmission condition (4.3) gives

$$b\left(\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix}\right) = \langle v_1, \overline{\psi_1}\rangle_\Gamma + \frac{1}{\mu}\langle v_0, \overline{\psi_0}\rangle_\Gamma - \frac{1}{\mu}\langle v_2, \overline{\psi_2} \rangle_1.$$

The three terms on the right-hand side are rewritten by using Green's formula (3.4)

$$\langle v_1, \overline{\psi_1}\rangle_\Gamma = \int_{\Omega_1} |\nabla u_1|^2\,dx - \overline{k_1^2}\int_{\Omega_1} |u_1|^2\,dx;$$

$$\langle v_0, \overline{\psi_0}\rangle_\Gamma = \int_{\Omega_1} |\nabla u_0|^2\,dx - \overline{k_2^2}\int_{\Omega_1} |u_0|^2\,dx;$$

$$-\langle v_2, \overline{\psi_2}\rangle_\Gamma = \int_{\Omega_2\cap B_R} |\nabla u_2|^2\,dx - \overline{k_2^2}\int_{\Omega_2\cap B_R} |u_2|^2\,dx - \int_{S_R} u_2\,\overline{\frac{\partial u_2}{\partial n}}\,ds,$$

where B_R is some large ball containing Ω_1,

If we take into account the representation formula (3.8), we see that the integrals

$$\int_{\Omega_1} |u_1|^2\,dx, \qquad \int_{\Omega_1} |u_0|^2\,dx, \qquad \int_{\Omega_2\cap B_R} |u_2|^2\,dx,$$

and

$$\int_{\Omega_R} u_2\,\overline{\frac{\partial u_2}{\partial n}}\,ds$$

are given by bilinear forms in the respective Cauchy data which are compact on $H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)$.

If we eliminate $\binom{v_2}{\psi_2}$ by the transmission condition, we obtain

$$\begin{aligned}
&\operatorname{Re} b\left(\begin{pmatrix} v_0\\ \psi_0\end{pmatrix}, \begin{pmatrix} v_1\\ \psi_1\end{pmatrix}\right) \qquad (5.11)\\
&\quad\geqslant \|u_1\|^2_{H^1(\Omega_1)} + \|u_0\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2\cap B_R)} - b_1\begin{pmatrix} v_1\\ \psi_1\end{pmatrix} - b_2\begin{pmatrix} v_0\\ \psi_0\end{pmatrix}\\
&\quad\geqslant \gamma(\|v_1\|^2_{1/2} + \|\psi_1\|^2_{-1/2} + \|v_0\|^2_{1/2} + \|\psi_0\|^2_{-1/2}) - b_1\begin{pmatrix} v_1\\ \psi_1\end{pmatrix} - b_2\begin{pmatrix} \psi_0\\ \psi_0\end{pmatrix},
\end{aligned}$$

where $\gamma > 0$ is a constant arising from the trace lemma, and b_1, b_2 are compact quadratic forms.

Now we assume that $\binom{v}{\psi}$ is a solution of the integral equation (4.5): $H\binom{v}{\psi} = \frac{1}{2}M^{-1}(1 + A_2)\binom{v_0}{\psi_0}$. It is no restriction to assume that $\binom{v_0}{\psi_0}$ satisfies (4.4). If not, we replace $\binom{v_0}{\psi_0}$ by $\frac{1}{2}(1 + A_2)\binom{v_0}{\psi_0}$. Furthermore we assume that $\binom{v}{\psi}$ also coincides with $\binom{v_1}{\psi_1}$ belonging to a solution of the transmission problem (this is implied by assumption $(\tilde{\mathrm{A}})$, for example). Then we have

$$\begin{pmatrix} \mathscr{D}v + \mathscr{K}'\psi \\ -\mathscr{K}v + \mathscr{V}\psi \end{pmatrix} = \mathscr{H}\begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu}\psi_0 \\ \\ v_0 \end{pmatrix},$$

hence from (5.10) follows

$$b\left(\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix}\right) = \langle -\mathscr{K}v + \mathscr{V}\psi, \bar{\psi}\rangle_\Gamma + \langle v, \overline{\mathscr{D}v + \mathscr{K}'\psi}\rangle_\Gamma$$
$$= \overline{\langle \mathscr{D}v + \mathscr{K}'\psi, \bar{v}\rangle_\Gamma} + \langle -\mathscr{K}v + \mathscr{V}\psi, \bar{\psi}\rangle_\Gamma.$$

From Definition 5.1 we see

$$\operatorname{Re} b\left(\begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} v_1 \\ \psi_1 \end{pmatrix}\right) = \operatorname{Re} a\left(\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\psi} \end{pmatrix}\right),$$

hence

$$\operatorname{Re} a\left(\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} \bar{v} \\ \bar{\psi} \end{pmatrix}\right) \geqslant \gamma(\|v\|_{1/2}^2 + \|\psi\|_{-1/2}^2) - b_1\begin{pmatrix} v \\ \psi \end{pmatrix} - b_2\left(2MH\begin{pmatrix} v \\ \psi \end{pmatrix}\right).$$

Thus we find Gårding's inequality again, but with this method it can be proved only for $\binom{v}{\psi}$ satisfying $(1-A_1)\binom{v}{\psi}=0$, and not for all $\binom{v}{\psi} \in H^{-1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$, as we did in Theorem 5.3. The latter has to be used for convergence proofs of Galerkin approximations (see Sect. 7), where we use the bilinear form a on the whole space $H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$.

6. Regularity of the Solutions of the Boundary Integral Equations

If the boundary Γ is smooth, system (4.5) of boundary integral equations is an elliptic system of pseudodifferential equations. The standard regularity theory for pseudodifferential operators shows that for given $\binom{v_0}{\psi_0} \in H^s(\Gamma) \oplus H^{s-1}(\Gamma)$, any solution $\binom{v}{\psi}$ of (4.5) is contained in $H^s(\Gamma) \oplus H^{s-1}(\Gamma)$. This valid for any $s \in \mathbb{R}$.

For nonsmooth Γ, this is not true due to the singularities at the corner points. We consider the case $n=2$, Γ a polygon, in this paragraph. For the solution of the transmission problem (2.1), (2.2), the corner singularities can be determined with Kondratiev's [32] method. This was done by Weisel [48] (compare [29]) for the case of Laplace's equation, i.e., $k_1 = k_2 = 0$. But Kondratiev's work shows that at least the first singularities are the same also for $k_j \neq 0$ and even for curved polygons. The resulting form of the solution implies for the Cauchy data on Γ the following decomposition:

Let $\binom{v_0}{\psi_0} \in H^s(\Gamma) \oplus \mathscr{H}^{s-1}(\Gamma)$ $(s \geqslant \frac{1}{2})$ be given and $\binom{v}{\psi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ be the Cauchy data of $u_1 \in \mathscr{L}_1$, solution of (2.1), (2.2). Then

$$\begin{pmatrix} v \\ \psi \end{pmatrix} = \sum_{j=1}^{J} \sum_{\ell=1}^{L_j^s} c_{j\ell} \begin{pmatrix} v_{j\ell} \\ \psi_{j\ell} \end{pmatrix} + \begin{pmatrix} v_s \\ \psi_s \end{pmatrix} \qquad \text{with } \begin{pmatrix} v_s \\ \psi_s \end{pmatrix} \in H^s(\Gamma) \oplus \mathscr{H}^{s-1}(\Gamma), \tag{6.1}$$

where the $v_{j\ell}(z)$ are on Γ^j and Γ^{j+1} of the form

$$d^{j\alpha r} \chi_j(z) \, |z - z_j|^{\alpha} \log^r |z - z_j|,$$

and similarly the $\psi_j(z)$ are of the form

$$\delta^{j\alpha r} \chi_j(z) \, |z - z_j|^{\alpha - 1} \log^{r'} |z - z_j|.$$

Here $\chi_j \in C_0^\infty(\mathbb{R}^2)$ are cut-off functions near the corner point z_j, the $d^{j\alpha r}$ and $\delta^{j\alpha r}$ are certain complex constants, possibly different for Γ^j and Γ^{j+1}, r, $r' \in \{0, 1, 2\}$, and $\alpha \in A_j$ with $0 < \operatorname{Re} \alpha < s - \frac{1}{2}$ for

$$A_j = \left\{ \alpha \in \mathbb{C} \;\middle|\; \left(\frac{\sin(\pi - \omega_j)\alpha}{\sin \pi\alpha} \right)^2 = \left(\frac{\mu + 1}{\mu - 1} \right)^2 \right\} \cup \mathbb{N}. \tag{6.2}$$

The $v_{j\ell}$ and $\psi_{j\ell}$ depend only on the geometry of the domain near z_j and not on $\binom{v_0}{\psi_0}$, whereas the constants c_{jl} and the smooth part $\binom{v_s}{\psi_s}$ depend on $\binom{v_0}{\psi_0}$.

We write $\mathscr{X}^s$ for the subspace of $H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ of all $\binom{v}{\psi}$ possessing a decomposition (6.1). $\mathscr{X}^s$ is a Hilbert space with the norm

$$\left\| \begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_{\mathscr{X}^s}^2 := \|v_s\|_s^2 + \|\psi_s\|_{\mathscr{H}^{s-1}}^2 + \sum_{j=1}^{J} \sum_{\ell=1}^{L_j} |c_{j\ell}|^2, \tag{6.3}$$

where

$$\|\psi_s\|_{\mathscr{H}^{s-1}}^s = \sum_{j=1}^{J} \|\psi_s|_{\Gamma^j}\|_{H^{s-1}(\Gamma^j)}^2.$$

Let us introduce the convention that in any place where we write $\|\cdot\|_{\mathscr{X}^s}$, we automatically include the condition $s - \frac{1}{2} \neq \operatorname{Re} \alpha$ for all $\alpha \in \bigcup_{j=1}^{J} A_j$. It is our aim to prove the decomposition (6.1) for the solutions of the integral equations (4.5) by using the method of local Mellin transformation for the integral operators. The result is

THEOREM 6.1. *Let Γ be a plane polygon and $\binom{v}{\psi} \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ a solution of*

$$H \begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ \phi \end{pmatrix} \qquad \textit{with } \begin{pmatrix} f \\ \phi \end{pmatrix} \in H^s(\Gamma) \oplus \mathscr{H}^{s-1}(\Gamma) \tag{6.4}$$

for some $s>\frac{1}{2}$, $s-\frac{1}{2}\notin \operatorname{Re}\bigcup_{j=1}^{J} A_j$ with A_j defined in (6.2). *Suppose further that either $k_1=k_2=0$ or $s<\frac{5}{2}$. Then $\binom{v}{\psi}\in\mathscr{X}^s$ and there is an a priori estimate*

$$\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_{\mathscr{X}^s}^2 \leqslant C\left\{\|f\|_s^2+\|\phi\|_{\mathscr{H}^{s-1}}^2+\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_{\mathscr{X}^{1/2}}^2\right\}. \tag{6.5}$$

Proof. Mellin transformation is possible only for the principal part H_0 of the operator H which is defined by the decomposition

$$H=H_0+C_0 \qquad \text{as in (5.3), i.e.,}$$

$$H_0=\begin{pmatrix} -K & \frac{1}{2}(1+\mu)V\\ \frac{1}{2}(1+1/\mu)D & K'\end{pmatrix}, \qquad C_0=\begin{pmatrix} -\mathscr{K}^1 & \mathscr{V}^1\\ \mathscr{D}^1 & \mathscr{K}'^1\end{pmatrix}.$$

From (5.7) we know that for $s<\frac{5}{2}$

$$C_0\colon H^{1/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\to H^s(\Gamma)\oplus\mathscr{H}^{s-1}(\Gamma),$$

hence we can write the integral equation (6.4) as

$$H_0\begin{pmatrix} v\\ \psi\end{pmatrix}=\begin{pmatrix} f'\\ \phi'\end{pmatrix}=\begin{pmatrix} f\\ \phi\end{pmatrix}-C_0\begin{pmatrix} v\\ \psi\end{pmatrix}\in H^s(\Gamma)\oplus\mathscr{H}^{s-1}(\Gamma) \qquad \text{for } s<\frac{5}{2}.$$

For $k_1=k_2=0$ we have $C_0=0$. Therefore we always may assume $k_1=k_2=0$ and $H=H_0$.

Now for H_0 we apply the method of Mellin transformation as developed in [10, 11, 13, 14]. One has to perform the following steps:

First the operator H_0 is considered on an infinite angle Γ^ω, which locally corresponds to Γ at the corner z_j with angle $\omega=\omega_j$, and H_0 is decomposed into H_1+H_2, where H_1 consists of multiplicative convolutions and H_2 is finite-dimensional.

Then H_1 is converted via Mellin transformation into an operator of multiplication by a meromorphic (4×4)-matrix valued function $\hat{H}(\lambda)$, the "Mellin symbol" of H.

Finally, the singular parts of the expansion (6.1) are found by determining the poles of the meromorphic function $\hat{H}(\lambda)^{-1}$ in the strip $\operatorname{Im}\lambda\in(0, s-\frac{1}{2})$.

More precisely, we proceed as follows:

Let $\Gamma^\omega=\Gamma^-\cup\{0\}\cup\Gamma^+$ with $\Gamma^-=e^{i\omega}\mathbb{R}_+$ and $\Gamma^+=\mathbb{R}_+$ $(\omega\in(0,2\pi))$. A function u on Γ^ω can be identified with the pair (u_-, u_+) of functions on $\mathbb{R}_+$ defined by $u_-(x)=u(xe^{i\omega})$; $u_+(x)=u(x)$ $(x>0)$. We will choose the representation of u by its even and odd parts, which are defined by

$$u^e(x)=\tfrac{1}{2}(u_-(x)+u_+(x)), \qquad u^o(x)=\tfrac{1}{2}(u_-(x)-u_+(x)).$$

This induces for any operator A acting on functions on Γ^ω a representation by a (2×2)-matrix of operators acting on functions on $\mathbb{R}_+$:

$$A \triangleq \mathbf{A} := \begin{pmatrix} A_{ee} & A_{oe} \\ A_{eo} & A_{oo} \end{pmatrix}, \qquad \text{where} \quad \begin{array}{l} (Au)^e = A_{ee}u^e + A_{oe}u^o \\ (Au)^o A_{eo}u^e + A_{oo}u^o. \end{array}$$

We need the following operators acting on functions on $\mathbb{R}_+$:

$$V_\omega\phi(x) := -\frac{1}{\pi}\int_0^\infty \log\left|1-\frac{x}{y}e^{-i\omega}\right| \phi(y)\,dy; \qquad V_0 = V_\omega \text{ for } \omega=0;$$

$$K_\omega\phi(x) := \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{1}{xe^{i\omega}-y}\right)\phi(y)\,dy;$$

$$K'_\omega\phi(x) := \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{e^{i\omega}}{xe^{i\omega}-y}\right)\phi(y)\,dy;$$

$$D_\omega\phi(x) := -\frac{1}{x}\frac{\partial}{\partial\omega}K_\omega\phi(x); \qquad D_0 = \lim_{\omega\to 0} D_\omega.$$

Then we obtain the following representation for $H_1 = H_0 - H_2$, where H_2 is some finite-dimensional operator which may be neglected:

$$\mathbf{H}_1 = \begin{pmatrix} -\mathbf{K} & \frac{1}{2}(1+\mu)\mathbf{V} \\ \frac{1}{2}(1+1/\mu)\mathbf{D} & \mathbf{K}' \end{pmatrix}$$

with

$$\mathbf{V} = \begin{pmatrix} V_0 + V_\omega & 0 \\ 0 & V_0 - V_\omega \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} K_0 & 0 \\ 0 & -K_\omega \end{pmatrix} \text{ (see [10])};$$

$$\mathbf{D} = \begin{pmatrix} D_\omega - D_0 & 0 \\ 0 & -(D_0 + D_\omega) \end{pmatrix} \text{ (see [13])}; \qquad \mathbf{K}' = \begin{pmatrix} K'_\omega & 0 \\ 0 & -K'_\omega \end{pmatrix}.$$

We know that the Mellin transformation defined by

$$\hat\phi(\lambda) := \int_0^\infty x^{i\lambda-1}\phi(x)\,dy$$

acts on these operators as follows (for $\phi \in C_0^\infty(0,\infty)$):

$$\widehat{V_\omega\phi}(\lambda) = \hat V_\omega(\lambda)\,\hat\phi(\lambda-i) \quad := \frac{\cosh(\pi-\omega)\lambda}{\sinh \pi\lambda}\,\hat\phi(\lambda-i) \quad (\operatorname{Im}\lambda \in (0,1));$$

$$\widehat{K_\omega\phi}(\lambda) = \hat K_\omega(\lambda)\,\hat\phi(\lambda) \quad := -\frac{\sinh(\pi-\omega)\lambda}{\sinh \pi\lambda}\,\hat\phi(\lambda) \quad (\operatorname{Im}\lambda \in (-1,1));$$

$$\widehat{K'_\omega \phi}(\lambda) = \hat{K}_\omega(\lambda + i)\,\hat{\phi}(\lambda) \qquad (\operatorname{Im}\lambda \in (-2, 0));$$

$$\widehat{D_\omega \phi}(\lambda) = \hat{D}_\omega(\lambda + i)\,\hat{\phi}(\lambda + i) := -(\lambda + i)\,\frac{\cosh(\pi - \omega)(\lambda + i)}{\sinh \pi(\lambda + i)}\,\hat{\phi}(\lambda + i)$$

$$\operatorname{Im}\lambda \in (-2, 0)).$$

Note that $\hat{D}_\omega(\lambda) = -\lambda^2 \hat{V}_\omega(\lambda)$.

Now we define

$$U = \begin{pmatrix} v^e \\ v^o \\ \psi^e \\ \psi^o \end{pmatrix} \qquad F = \begin{pmatrix} f^e \\ f^o \\ \phi^e \\ \phi^o \end{pmatrix}$$

so that the equation $H_1\binom{v}{\psi} = \binom{f}{\phi}$ on Γ^ω is equivalent to $\mathbf{H}_1 U = F$.

Mellin transformation yields the equation

$$\hat{H}(\lambda)\,\hat{U}(\lambda) = \hat{F}(\lambda) \tag{6.6}$$

with

$$\hat{H}(\lambda) = \begin{pmatrix} -\hat{K}_\omega(\lambda) & 0 & \frac{1}{2}(1+\mu)(\hat{V}_\omega(\lambda) + \hat{V}_0(\lambda)) & 0 \\ 0 & \hat{K}_\omega(\lambda) & 0 & \frac{1}{2}(1+\mu)(\hat{V}_0(\lambda) - \hat{V}_\omega(\lambda)) \\ \frac{1}{2}(1 + 1/\mu)(\hat{D}_\omega(\lambda) - \hat{D}_0(\lambda)) & 0 & \hat{K}_\omega(\lambda) & 0 \\ 0 & -\frac{1}{2}(1 + 1/\mu)(\hat{D}_\omega(\lambda) + \hat{D}_0(\lambda)) & 0 & -\hat{K}_\omega(\lambda) \end{pmatrix}$$

and

$$\hat{U}(\lambda) = \begin{pmatrix} \hat{v}^e(\lambda) \\ \hat{v}^o(\lambda) \\ \hat{\psi}^e(\lambda - i) \\ \hat{\psi}^o(\lambda - i) \end{pmatrix}; \qquad \hat{F}(\lambda) = \begin{pmatrix} \hat{f}^e(\lambda) \\ \hat{f}^o(\lambda) \\ \hat{\phi}^e(\lambda - i) \\ \hat{\phi}^o(\lambda - i) \end{pmatrix}.$$

Now we have to assume that $\hat{F}$ is meromorphic with simple poles $\lambda \in \{0, i, 2i, \ldots\}$ and to find the poles of $\hat{U}$ from Eq. (6.6). A pole of order $r_0 + 1$ at $\lambda = i\alpha$ will correspond to contributions of the form $|z - z_j|^\alpha \log^r |z - z_j|$ $(r = 0, 1, \ldots, r_0)$ for v, and of the form $|z - z_j|^{\alpha - 1} \log^{r'} |z - z_j|$

($r'=0, 1,\dots, r_0$) for ψ. The constants $d^{j\alpha r}$ and $\delta^{j\alpha r}$ can be determined from the residues of $\hat{U}(\lambda)$ at $\lambda = i\alpha$. We will not calculate the residues of $\hat{U}(\lambda)$ here (see [10] for examples) but only find the poles of $\hat{U}$. They correspond to poles of $\hat{H}(\lambda)^{-1}$.

From the equation $\hat{U}(\lambda)=\hat{H}^{-1}(\lambda)\,\hat{F}(\lambda)$ for $\operatorname{Im}\lambda = s-\frac{1}{2}$ one also (locally) finds the regular part $\binom{v_s}{\psi_s}\in H^s(\Gamma)\oplus\mathscr{H}^{s-1}(\Gamma)$ and the a priori estimate (6.5). (See [10, 14].)

Now it only remains to determine the zeroes of $\det\hat{H}(\lambda)$. From the special form of $\hat{H}(\lambda)$ it follows easily that

$$\det\hat{H}(\lambda)=\Delta^e(\lambda)\cdot\Delta^o(\lambda)$$

with

$$\Delta^e(\lambda)= -\hat{K}_\omega(\lambda)^2-\frac{(\mu+1)^2}{4\mu}\,(\hat{V}_\omega(\lambda)+\hat{V}_0(\lambda))(\hat{D}_\omega(\lambda)-\hat{D}_0(\lambda));$$

$$\Delta^o(\lambda)= -\hat{K}_\omega(\lambda)^2-\frac{(\mu+1)^2}{4\mu}\,(\hat{V}_\omega(\lambda)-\hat{V}_0(\lambda))(\hat{D}_\omega(\lambda)+\hat{D}_0(\lambda)).$$

We see that $\Delta^o=\Delta^e$, hence it suffices to look at the equation

$$\Delta^o(\lambda)=0. \tag{6.7}$$

We obtain

$$\begin{aligned}\Delta^o(\lambda)&=\frac{-1}{\sinh^2\pi\lambda}\cdot\frac{1}{4\mu}\,\{4\mu\sinh^2\pi\lambda+(\mu-1)^2\sinh(2\pi-\omega)\lambda\cdot\sinh\omega\lambda\}\\&=\frac{1}{\sinh^2\pi\lambda}\cdot\frac{(\mu+1)^2}{4\mu}\left\{-\sinh^2\pi\lambda+\left(\frac{\mu-1}{\mu+1}\right)^2\sin^2(\pi-\omega)\lambda\right\}.\end{aligned}$$

The first form of Δ^o shows that (6.7) for $\lambda=i\alpha$ yields the transcendental equation obtained by Weisel [48] for the exponents of the singular functions. The second form shows (6.7) for $\lambda=i\alpha$ has the solution sets A_j as defined in (6.2) (for $\omega=\omega_j$). Here we always supposed $\mu\neq -1$ and $\alpha\notin\mathbb{N}_0$. ∎

The equation (6.7) can be used to determine the dependence of the singularities of the solution on the transmission coefficient μ. We will do this for the exponent of the first singular function in the expansion (6.1). One can also study the dependence of the constants in (6.1) on μ by calculating the residues of $\hat{U}$ from (6.6).

We write (6.7) in the form

$$\mu^{\pm 1} \tan \frac{\omega\alpha}{2} = -\tan(2\pi - \omega)\, \frac{\alpha}{2} \qquad \left(= \tan\left(\frac{\omega\alpha}{2} + \pi(1-\alpha)\right)\right), \quad (6.8)$$

where both signs in $\mu^{\pm 1}$ give solutions $\lambda = i\alpha$ of (6.7).

The following lemma shows that the solution of the transmission problem (2.1), (2.2) for any $\mu > 0$, $\mu \neq 1$ and any angle $\omega \neq \pi$ is in general not contained in $H^2_{\mathrm{loc}}(\Omega_j)$ $(j = 1, 2)$:

LEMMA 6.2. *Let $\mu > 0$, $\mu \neq 1$, and $\omega \neq \pi$ and define*

$$\omega' = \min\{\omega, 2\pi - \omega\} \in (0, \pi); \qquad \alpha_1 = \frac{\pi}{2\pi - \omega'}; \qquad \alpha_2 = \min\left\{\frac{\pi}{\omega'}, \frac{2\pi}{2\pi - \omega'}\right\}.$$

Then

(i) *the solution λ_0 of (6.7) with minimal positive imaginary part is purely imaginary: $\lambda_0 = i\alpha_0$.*

(ii) $\alpha_0 \in (\alpha_1, 1) \subset (\frac{1}{2}, 1)$.

(iii) *The equation (6.8), i.e., (6.7) for $\lambda = i\alpha$, has in (α_1, α_2) exactly two real solutions $\alpha_0 < 1 < \alpha^0$, and there holds $\alpha_0 \to \alpha_1$ and $\alpha^0 \to \alpha_2$ if μ tends to zero or to infinity.*

Proof. We may assume $\omega = \omega'$. Consider real solutions $\alpha \in (0, \alpha_2)$ of the equation

$$f_\mu(\alpha) := \mu \tan \frac{\omega\alpha}{2} + \tan(2\pi - \omega)\, \frac{\alpha}{2} = 0.$$

Now $\mu \tan(\omega\alpha/2) > 0$ implies $\alpha \in (\pi/(2\pi - \omega), 2\pi/(2\pi - \omega))$; and for $\alpha \searrow \alpha_1 = \pi/(2\pi - \omega)$ we have $f_\mu(\alpha) \to -\infty$, whereas for $\alpha \nearrow \alpha_2$ we have $f_\mu(\alpha) \to \mu \tan(\omega\alpha_2/2) > 0$ if $\alpha_2 = 2\pi/(2\pi - \omega)$, i.e., for $\omega > 2\pi/3$, and $f_\mu(\alpha) \to +\infty$ if $\alpha_2 = \pi/\omega$, i.e., $\omega \leqslant 2\pi/3$. The strict monotonicity of f implies that there is exactly one zero $\alpha_+ \in (\alpha_1, \alpha_2)$. Writing $f_\mu(\alpha) = \mu \tan(\omega\alpha/2) - \tan(\omega\alpha/2 + \pi(1-\alpha))$, we see that

$$\alpha_+ < 1 \text{ for } \mu > 1 \qquad \text{and} \qquad \alpha_+ > 1 \text{ for } \mu < 1$$

and also

$$\alpha_+ \searrow \alpha_1 \text{ for } \mu \to \infty, \qquad \alpha_+ \nearrow \alpha_2 \text{ for } \mu \to 0.$$

Let $\alpha_- \in (\alpha_1, \alpha_2)$ be the solution of $f_{1/\mu}(\alpha) = 0$ and define $\alpha_0 = \min\{\alpha_+, \alpha_-\}$, $\alpha^0 = \max\{\alpha_+, \alpha_-\}$. Then α_0 and α^0 have the properties stated in (ii) and (iii).

Now we prove (i): Suppose $\lambda = i(\alpha + i\beta)$, $\alpha, \beta > 0$, $\alpha \in (0, 1)$. Taking real and imaginary parts of the equation

$$\sinh \pi\lambda = \sigma \frac{\mu - 1}{\mu + 1} \sinh(\pi - \omega)\lambda \qquad (\sigma = \pm 1),$$

we obtain

$$\sin \pi\alpha = \sigma \frac{\mu - 1}{\mu + 1} \frac{\cosh(\pi - \omega)\beta}{\cosh \pi\beta} \sin(\pi - \omega)$$

and (6.9)

$$\frac{\sinh \pi\beta}{\sinh(\pi - \omega)\beta} = \sigma \frac{\mu - 1}{\mu + 1} \frac{\cos(\pi - \omega)\alpha}{\cos \pi\alpha}.$$

If we choose $\tilde{\mu} > 0$ such that $(\tilde{\mu} - 1)/(\tilde{\mu} + 1) = (\mu - 1)/(\mu + 1)$ $\cosh(\pi - \omega)\beta/\cosh \pi\beta$ holds, we see that α has to be a zero of $f_{\tilde{\mu}}(\alpha)$. From the first part of the proof follows $\alpha > \alpha_1$.

From (6.9) follows

$$\frac{\tanh \pi\beta}{\tanh(\pi - \omega)\beta} = \frac{\tan \pi\alpha}{\tan(\pi - \omega)\alpha}. \tag{6.10}$$

Because of $0 < (\pi - \omega)\beta < \pi\beta$, the left-hand side in (6.10) is > 1. From $\pi\alpha \in (\pi\alpha_1, \pi) \subset (\pi/2, \pi)$ follows $\tan \pi\alpha < 0$, hence $\tan(\pi - \omega)\alpha < 0$, hence $(\pi - \omega)\alpha > \pi/2$. Thus $\frac{1}{2} < (\pi - \omega)\alpha < \pi\alpha < \pi$, hence $\tan(\pi - \omega)\alpha < \tan \pi\alpha < 0$, and therefore the right-hand side of (6.10) is < 1. This contradiction shows that there is no such solution with $\beta \neq 0$. Then $\lambda_0 := i\alpha_0$ is the solution with minimal positive imaginary part because it is the only solution with $\operatorname{Im} \lambda \in (0, 1)$. ∎

7. Galerkin Approximation for the Boundary Integral Equations

The Gårding inequality derived in Section 5 together with uniqueness yield quasioptimal error estimates for any Galerkin approximation procedure for the system (4.5) of boundary integral equations defined by means of the bilinear form a given in Definition 5.1.

For a smooth boundary Γ, this implies asymptotic error estimates for finite element approximation schemes which are known for any strongly elliptic system of pseudodifferential operators [46, 50, 51, 52, 24]. See Proposition 7.2.

In this paragraph, we shall concentrate on the case of a plane polygon Γ

and give asymptotic error estimates for the Fix method. From now on we assume that both assumptions (A) and ($\tilde{\mathrm{A}}$) hold, i.e., the operator H is bijective.

A general Galerkin procedure involves a family of finite dimensional subspaces $S_h \subset H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$ $(h \in (0, h_0))$ such that $\bigcup_{h>0} S_h$ is dense in $H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$, and the problem:

Find $U^h = \binom{v^h}{\psi^h} \in S_h$ such that

$$a(U^h, W) = a(U, W) \qquad \text{for all } W \in S_h, \tag{7.1}$$

where $U = \binom{v}{\psi}$ is the exact solution of the system

$$HU = \frac{1}{2} M^{-1}(1 + A_2) \begin{pmatrix} v_0 \\ \psi_0 \end{pmatrix} =: F = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{7.2}$$

By definition, (7.1) means

$$\langle \mathscr{D} v^h + \mathscr{K}' \psi^h, w \rangle_\Gamma + \langle -\mathscr{K} v^h + \mathscr{V} \psi^h, \chi \rangle_\Gamma = \langle \phi, w \rangle_\Gamma + \langle f, \chi \rangle_\Gamma$$
$$\text{for all } W = \begin{pmatrix} w \\ \chi \end{pmatrix} \in S_h. \tag{7.3}$$

Now Gårding's inequality, Theorem 5.3, and the invertibility of the operator $\mathscr{H}$, Corollary 4.5, together imply the following quasioptimal error estimate for the Galerkin procedure (7.1) by standard arguments [25, 10].

PROPOSITION 7.1. *There exists $h_0 > 0$ such that for any $h \in (0, h_0)$, the Galerkin equations* (7.1) *have a unique solution $U^h \in S_h$, and there exists $C > 0$ such that for the exact solution $U \in H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma) = \mathscr{X}^{1/2}$ of* (7.2) *there holds*

$$\|U - U^h\|_{\mathscr{X}^{1/2}} \leqslant C \inf_{W \in S_h} \|U - W\|_{\mathscr{X}^{1/2}}. \tag{7.4}$$

In order to obtain rates of convergence for $U - U^h$, one has to make assumptions on the approximation of U on the right-hand side of (7.4). For a smooth boundary, we may take the $S^{t,k}$-systems [3]:

We define

$$S_h := S_h^{t,k}(\Gamma) \oplus S_h^{t^*,k-1}(\Gamma)$$
$$(t^* = \max\{1, t-1\},\ t \in \mathbb{N},\ k \in \mathbb{N}_0,\ t > k). \tag{7.5}$$

These spaces satisfy the conformity condition $S_h \subset H^k(\Gamma) \oplus H^{k-1}(\Gamma)$. Furthermore, they have the well-known approximation property yielding asymptotic estimates for the error $U - U^h$ in $H^{1/2}(\Gamma) \oplus H^{-1/2}(\Gamma)$; they satisfy the inverse assumption yielding estimates in $H^s(\Gamma) \oplus H^{s-1/2}(\Gamma)$ for

$s > \frac{1}{2}$; and from the symmetry of H (Lemma 4.4) we see that the Aubin–Nitsche lemma [28] can be applied to yield such estimates also for $s < \frac{1}{2}$. We quote from [28] the following result:

PROPOSITION 7.2. *Let Γ be smooth, $-t+1 \leqslant s \leqslant r < t + \frac{1}{2}$, $s \leqslant k$. Then there is a $C > 0$ such that for any $h \in (0, h_0)$*

$$\|U - U^h\|_{H^s(\Gamma) \oplus H^{s-1}(\Gamma)} \leqslant Ch^{r-s} \|U\|_{H^r(\Gamma) \oplus H^{r-1}(\Gamma)} \leqslant Ch^{r-s} \|F\|_{H^r(\Gamma) \oplus H^{r-1}(\Gamma)}. \tag{7.6}$$

We note that from any estimate on the boundary there follow estimates for the corresponding approximate solution u_j^h of the transmission problem (2.1)–(2.5) in the domains Ω_j $(j = 1, 2)$. Here u_j^h is defined by means of the representation formula (3.8) applied to the Galerkin solution U^h in the same way as the exact solution u_j is obtained from U. Recall that $U = \binom{v}{\psi}$ are the Cauchy data of u_1, and $\binom{v - v_0}{\mu\psi - \psi_0}$ are those of u_2.

As an example, for any $\chi \in C_0^\infty(\mathbb{R}^n)$ there is an estimate

$$\|\chi(u_j - u_j^h)_{H^{s+1/2}(\Omega_j)} \leqslant C \|U - U^h\|_{H^s(\Gamma) \oplus H^{s-1}(\Gamma)}$$

with $C > 0$ independent of $h \in (0, h_0)$ (for any $s \in \mathbb{R}$, if Γ is smooth).

Furthermore, in any compact part Ω' of Ω_j, one obtains estimates in any norm, e.g.,

$$\|u_j - u_j^h\|_{L^\infty(\Omega')} \leqslant Ch^{2t-1} \|F\|_{H^t(\Gamma) \oplus H^{t-1}(\Gamma)}. \tag{7.7}$$

Thus in (7.6) and in (7.7) we can obtain arbitrary high convergence rates by choosing the smoothness of the right-hand side F and the degree $t-1$ of the approximating piecewise polynomials high enough. In the case of a scattering problem, F is given by the incident field which is smooth and gives $F \in C^\infty(\Gamma)$ (compare (2.6)).

For the case of a polygonal boundary $\Gamma \subset \mathbb{R}^2$ this is different: Even for $F \in C^\infty(\Gamma)$ one finds only (see Theorem 6.1 and Lemma 6.2)

$$U \in H^{\alpha_0 + (1/2) - \varepsilon}(\Gamma) \oplus H^{\alpha_0 - (1/2) - \varepsilon}(\Gamma)$$

where $\varepsilon > 0$ and

$$\alpha_0 = \min \left\{ \operatorname{Re} \alpha \mid \operatorname{Re} \alpha > 0,\ \alpha \in \bigcup_{j=1}^{J} A_j \right\} \in \left(\frac{1}{2}, 1 \right).$$

Therefore in (7.6) one can choose only $r \leqslant \alpha_0 + \frac{1}{2} - \varepsilon$ which, e.g., for $s = \frac{1}{2}$, i.e., the energy norm, gives a convergence rate $h^{1/2}$ but in general no h^σ with $\sigma > \frac{1}{2}$.

In order to obtain higher convergence rates, according to the Fix method one includes the explicitly known singular functions $U_j := \binom{v_{j\ell}}{\psi_{j\ell}}$ from the expansion (6.1) into the spaces S_h. Thus only the smooth part $\binom{v_s}{\psi_s}$ in (6.1) of U is approximated by piecewise polynomials. We define as in [10] the space $S_h^{p,t,k}$ by

$$\begin{aligned} &\tilde{U} \in S_h^{p,t,k} :\Leftrightarrow \tilde{U} = \tilde{U}_0 + \sum_{j=1}^{J} \sum_{\ell=1}^{L_j^p} \tilde{c}_{j\ell} U_{j\ell}, \\ &\tilde{U}_0 \in S_h^{t,k} \oplus \mathscr{S}_h^{t^*,k-1}. \end{aligned} \tag{7.8}$$

Here for $U_{j\ell}$ ($\ell = 1,..., L_j^p$) we take all functions $U_j = \binom{v_{j\ell}}{\psi_{j\ell}}$ which have on an arbitrary segment Γ^m, $m = 1,..., J$, the form

$$\begin{aligned} v_{j\ell}(z) &= \chi_j(z)\,|z - z_j|^{\alpha} \log^r |z - z_j|; \\ \psi_{j\ell}(z) &= \chi_j(z)\,|z - z_j|^{\alpha - 1} \log^{r'} |z - z_j|, \end{aligned} \tag{7.9}$$

where $r, r' = 0, 1, 2$; $\chi_j \in C_0^\infty(\mathbb{R}^2)$ with $\chi_j \equiv 1$ near z_j and $\Gamma \cap \operatorname{supp} \chi_j \subset (\Gamma^j \cup \{z_j\} \cup \Gamma^{j+1})$, and

$$\alpha \in A_j \qquad \text{with} \quad 0 < \operatorname{Re} \alpha < p - \tfrac{1}{2}. \tag{7.10}$$

$S_h^{t,k}$ as above denotes piecewise polynomials of degree $t-1$ satisfying compatibility conditions such that $S_h^{t,k} \subset H^k(\Gamma)$, in particular $t > k$. Correspondingly, we require $\mathscr{S}_h^{t^*,k-1} \subset \mathscr{H}^{k-1}(\Gamma)$, i.e.,

$$\tilde{\psi} \in \mathscr{S}_h^{t^*,k-1} \Leftrightarrow \tilde{\psi}\,|_{\Gamma^j} \in S_h^{t^*,k-1}(\Gamma^j) \qquad \text{for all } j = 1,..., J.$$

Thus the augmented finite element spaces satisfy the conformity condition

$$S_h^{p,t,k} \subset \mathscr{X}^q \qquad \text{for} \quad k \geqslant q \geqslant \tfrac{1}{2} \text{ and any } p. \tag{7.11}$$

They also have the *approximation property* [54, 10]:

Let $U \in \mathscr{X}^r$ and $p \geqslant r$ and $k \geqslant r$. Then to any $h \in (0, h_0)$ there exists $\tilde{U}_h \in S_h^{p,t,k}$ such that for all $\frac{1}{2} \leqslant q \leqslant r$ there holds

$$\|U - \tilde{U}_h\|_{\mathscr{X}^q} \leqslant C h^{r-q} \|U\|_{\mathscr{X}^r} \tag{7.12}$$

with a constant C depending neither on U nor on h.

Furthermore, the spaces $S_h^{p,t,k}$ satisfy the *inverse assumption* [54, 10]:

For $q \leqslant r \leqslant k$ and $\varepsilon > 0$ there exist $M > 0$ such that for all $h \in (0, h_0)$ and any $\tilde{U}_h \in S_h^{p,t,k}$ there holds

$$\|\tilde{U}_h\|_{\mathscr{X}^r} \leqslant M h^{q-r-\varepsilon} \|\tilde{U}_h\|_{\mathscr{X}^q} \tag{7.13}$$

where $\varepsilon = 0$ is allowed if $\operatorname{Re} \alpha \notin [q - \frac{1}{2}, r' - \frac{1}{2}]$ for all $\alpha \in A = \bigcup_{j=1}^{J} A_j$; A_j is given by (6.2), and $r' = \max\{p, r\}$.

Now we use the Gårding inequality, the bijectivity of H in $\mathscr{X}^{1/2}$ and the regularity result of Theorem 6.1 and obtain by standard arguments using (7.11)–(7.13)

THEOREM 7.3. *Let $\Gamma \subset \mathbb{R}^2$ be a polygon and $U \in \mathscr{X}^{1/2}$ solve the equation (7.2) $HU = F \in H^r(\Gamma) \oplus \mathscr{H}^{r-1}(\Gamma)$ where $r \geqslant \frac{1}{2}$, $r - \frac{1}{2} \neq \operatorname{Re} \alpha$ for all $\alpha \in A$, and either $k_1 = k_2 = 0$ or $r < \frac{5}{2}$. Then there exists $h_0 > 0$ such that to any $h \in (0, h_0)$ there is exactly one solution $U^h \in S_h^{p,t,k}$ of the Galerkin equations (7.1) with $S_h = S_h^{p,t,k}$ (to any given $p \in \mathbb{R}$, $t \in \mathbb{N}$, $k \in \mathbb{N}$ with $t > k$). Furthermore, there is a constant $C > 0$, independent of U and h, such that*

$$\|U - U^h\|_{\mathscr{X}^{1/2}} \leqslant C \inf_{W \in S_h^{p,t,k}} \|U - W\|_{\mathscr{X}^{1/2}}.$$

In addition, for $\frac{1}{2} \leqslant s \leqslant r \leqslant k$ and $p \geqslant r$ such that $\operatorname{Re} \alpha \notin \{r - \frac{1}{2}, s - \frac{1}{2}\}$ for all $\alpha \in A = \bigcup_{j=1}^{J} A_j$, and any $\varepsilon > 0$ there exists a constant C, independent of U and h, such that for $h \in (0, h_0)$

$$\|U - U^h\|_{\mathscr{X}^s} \leqslant C h^{r-s-\varepsilon} \|U\|_{\mathscr{X}^r} \leqslant C h^{r-s-\varepsilon} \|F\|_{H^r(\Gamma) \oplus \mathscr{H}^{r-1}(\Gamma)}. \quad (7.14)$$

Remark 7.4. (i) The spaces $S_h^{p,t,k}$, as defined in (7.8)–(7.10), may contain more singular functions than actually needed. By a careful study of Eq. (6.6) and computation of the residues of $\hat{U}(\lambda)$ at $\lambda = i\alpha$ one can derive relations between the constants $\hat{c}_{j\ell}$ in (7.8) and thereby reduce the numbers L_j^p (compare [10, Sect. 4]), and the definition of $\mathscr{X}^s$ and $S_h^{p,t,k}$ there. For example, in most cases the logarithmic terms, i.e., r, $r' > 0$, in (7.9) are not necessary.

(ii) The estimate (7.14) contains for $s > \operatorname{Re} \alpha + \frac{1}{2}$ also estimates for the coefficients c_j of the singular functions with exponent α, due to the definition (6.3) of the norm in $\mathscr{X}^s$.

We illustrate our results by an example: Let the L-shaped boundary $\Gamma \subset \mathbb{R}^2$ and the transmission problem be described by Fig. 1. Here, the angles are $\pi/2$ or $3\pi/2$, and from Lemma 6.2 we find

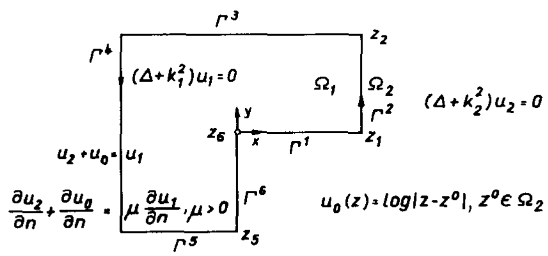

FIGURE 1

$$\frac{2}{3} < \alpha_0 = \frac{2}{\pi} \arctan \frac{3\mu^2 + 10\mu + 3}{(\mu - 1)^2} < 1, \qquad \alpha^0 = 2 - \alpha_0 \text{ (compare [48])}.$$

We include into the augmented finite element spaces the singular functions with the exponents α_0 and α^0. These are

$$u_{j\ell} = \begin{pmatrix} v_{j\ell} \\ \psi_{j\ell} \end{pmatrix}, \qquad \ell = 1, \ldots, 4,\ j = 1, \ldots, 6;$$

$$\begin{aligned}
v_{j1}(z) &= \chi_j(z)\, |z - z_j|^{\alpha_0} && \text{on } \Gamma^j, \\
&= 0 && \text{on } \Gamma^{j+1}, \\
v_{j2}(z) &= 0 && \text{on } \Gamma^j, \\
&= \chi_j(z)\, |z - z_j|^{\alpha_0} && \text{on } \Gamma^{j+1}, \\
v_{j3}(z) &= \chi_j(z)\, |z - z_j|^{\alpha^0} && \text{on } \Gamma^j, \\
&= 0 && \text{on } \Gamma^{j+1}, \\
v_{j4}(z) &= 0 && \text{on } \Gamma^j, \\
&= \chi_j(z)\, |z - z_j|^{\alpha^0} && \text{on } \Gamma^{j+1};
\end{aligned}$$

$\psi_{j\ell}$ are defined correspondingly with exponents $\alpha_0 - 1$ and $\alpha^0 - 1$. For the smooth parts v^h and ψ_h we use piecewise quadratic and piecewise linear polynomials, respectively.

The regularity theorem 6.1 give the decomposition

$$\begin{pmatrix} v \\ \psi \end{pmatrix} = \sum_{j=1}^{6} \sum_{\ell=1}^{4} \begin{pmatrix} c_{j\ell}\, v_{j\ell} \\ d_{j\ell} \psi_{j\ell} \end{pmatrix} + \begin{pmatrix} v_s \\ \psi_s \end{pmatrix},$$

$$\begin{pmatrix} v_s \\ \psi_s \end{pmatrix} \in H^s(\Gamma) \oplus H^{s-1}(\Gamma),\ s = \frac{11}{6},$$

and we use the corresponding notation for $U^h = \binom{v^h}{\psi^h}$. Then Theorem 7.3 yields the convergence rates

$$\|u - u^h\|_{\mathscr{H}^{1/2}} = O(h^{4/3 - \varepsilon});$$

$$\begin{aligned}
|c_{j\ell} - c_{j\ell}^h| &= O(h^{4/3 - \alpha_0 - \varepsilon}) = O(h^{1/3}); \\
|d_{j\ell} - d_{j\ell}^h| &= O(h^{4/3 - \alpha_0 - \varepsilon}) = O(h^{1/3}) && (\ell = 1, 2); \\
|c_{j\ell} - c_{j\ell}^h| &= O(h^{4/3 - \alpha^0 - \varepsilon}); \\
|d_{j\ell} - d_{j\ell}^h| &= O(h^{4/3 - \alpha^0 - \varepsilon}) && (\ell = 3, 4).
\end{aligned}$$

We make the following observations: Without singular test and trial functions there is a higher convergence rate for any finite μ than in the limit cases $\mu \to 0$ or $\mu \to \infty$ which correspond to interior or exterior Dirichlet or Neumann problems, because the solution of the transmission problem is more regular—$\alpha_0 > \frac{2}{3}$. Note that $\frac{2}{3}$ is the exponent of the first singular function at an angle $3\pi/2$ for the boundary value problems (compare [11]). On the other hand, by the same reason, the convergence rates of the coefficients of the first singular function are lower in the transmission problem than in the boundary value problems.

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