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Strongly elliptic boundary integral equations for electromagnetic transmission problems

Martin Costabel
Fachbereich Mathematik, Technische Hochschule Darmstadt, Schlossgartenstr. 7, 6100 Darmstadt, Germany

and

Ernst P. Stephan*
School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 330332, U.S.A.

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Synopsis

We study a boundary integral equation method for transmission problems for strongly elliptic differential operators, which yields a strongly elliptic system of pseudodifferential operators and which therefore can be used for numerical computations with Galerkin's procedure. The method is shown to work for the vector Helmholtz equation in $\mathbb{R}^3$ with electromagnetic transmission conditions. We propose a slightly modified system of boundary values in order for the corresponding bilinear form to be coercive over H^1. We analyse the boundary integral equations using the calculus of pseudodifferential operators. Here the concept of the principal symbol is used to derive existence and regularity results for the solution.

1. Introduction

The usefulness of strongly elliptic pseudodifferential operators for numerical methods is now well established ([**9**, **13**, **23**, **29**, **30**, **11**]). For boundary value problems, there are many papers dealing with the analytical and computational aspects of various examples demonstrating this fact. The so-called "direct method" leads easily from a strongly elliptic boundary value problem to a strongly elliptic system of pseudodifferential equations on the boundary. This method is now well understood and also has, by its simplicity, a wide range of applications to mixed boundary value problems and to problems with irregular boundaries [**3**]. This is one advantage of such first kind integral equations compared to the traditionally-preferred Fredholm integral equations of the second kind. For the latter equations, there is no generally applicable method of derivation, and they lose their Fredholm properties as soon as the coefficients or the boundaries are not smooth. Other advantages of the direct method are that the solutions have direct physical meaning, and the appropriate norm is the energy norm. Therefore the corresponding boundary element methods share several nice properties of the usual finite element methods [**31**].

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For acoustic transmission problems, the authors showed in [4] that for the case of the Helmholtz equation the direct method also leads to strongly elliptic boundary integral equations. This was done using the symbols of the operators defined by local Fourier transformation in the case of smooth boundaries in any dimension, respectively by local Mellin transformation in the case of a two-dimensional domain with corners (see also [26]).

In the present paper, we show how to derive strongly elliptic boundary integral equations for a general class of strongly elliptic transmission problems, by the direct method. We then apply this to electromagnetic transmission problems. To do this we must have a bilinear form, connected with the boundary data through Green's first theorem, that is, coercive over all of H^1, whereas the usual energy form is not [8]. To achieve this, we use a modification of the boundary data which gives rise to a transmission problem *equivalent* to the original one. Then we compute the kernels of the integral operators and their principal symbols which show clearly how the proposed modification transforms the system of operators into a strongly elliptic one. This kind of transformation of the boundary integral operators was first used by MacCamy and Stephan in [18] for the perfect conductor problem, i.e. the exterior boundary value problem for Maxwell's equations with given electric data.

Under the assumption of uniqueness we derive existence of the solution of our integral equation on the transmission manifold. Since the general electromagnetic transmission problem is equivalent to our boundary integral equation, we have an analytic solution procedure via the integral equation; in connection with Galerkin's method we even have a numerical solution procedure by boundary elements. Our system of operators for the transmission problem contains the systems which can be used for electric and magnetic boundary value problems and for screen scattering problems with electric and with magnetic boundary data. For all these cases we therefore have strong ellipticity of the corresponding boundary integral equations. For screen problems, these and corresponding symbols are the starting point of an analysis of singularities at the screen edge, see [25, 24].

2. The direct method for transmission problems

The "direct method" for strongly elliptic boundary value problems was studied in [5]. Here we present a related analysis of transmission problems. We see that the case of the scalar Helmholtz equation studied in [4] is representative of a general class of transmission problems. A similar analysis is possible for very general combinations of boundary and transmission conditions [21].

Let Ω_1 be a bounded domain in $\mathbb{R}^n$ with boundary $\Gamma \in C^\infty$ and $\Omega_2 := \mathbb{R}^n \setminus \overline{\Omega_1}$. We consider the following transmission problem:

$$P_1 u_1 = 0 \quad \text{in } \Omega_1, \tag{2.1}$$

$$P_2 u_2 = 0 \quad \text{in } \Omega_2, \tag{2.2}$$

u_2 satisfies some "radiation condition", see (2.17),

$$R_2\gamma_2 u_2 - R_1\gamma_1 u_1 = u_0 \quad \text{on } \Gamma. \tag{2.3}$$

Here, for $j = 1, 2$, P_j are elliptic differential operators of order $2m$ with C^∞

coefficients, for simplicity both defined throughout $\mathbb{R}^n$; $\gamma_j u_j$ are the Cauchy data of u_j on Γ from Ω_j:

$$\gamma_j u_j = (u_j, \partial_n u_j, \ldots, \partial_n^{2m-1} u_j)|_\Gamma, \tag{2.4}$$

where ∂_n means the normal derivative with respect to the normal pointing from Ω_1 to Ω_2; $R_j\gamma_j$ are systems of $2m$ differential operators with C^∞ coefficients:

$$(R_j\gamma_j u_j)_i = \sum_{k=0}^{2m-1} R_j^{ik}\, \partial_n^k u_j|_\Gamma, \tag{2.5}$$

where R_j^{ik} are tangential differential operators of order

$$\operatorname{ord} R_j^{ik} = i - k. \tag{2.6}$$

We assume that R_j are Dirichlet systems of order $2m$ (see [**17**]), which implies, as well as (2.6), that the lower triangular matrix

$$R_j = (R_j^{ik})_{i,k=0,\ldots,2m-1}$$

is invertible, the inverse also being a tangential differential operator.

In order to transform the transmission problem to a problem on the boundary, we must assume that we know fundamental solutions G_j for the differential operators P_j, i.e. two-sided inverses on the space $\mathscr{E}'$ of distributions with compact support on $\mathbb{R}^n$. This hypothesis immediately makes available a "second Green formula" and a "representation formula" as follows: Let

$$P_i = \sum_{l=0}^{2m} P_{jl}\, \partial_n^l \tag{2.7}$$

be the representation of P_j near Γ with differential operators P_{jl} of order $2m - l$ which are tangential on Γ. Let $u_j \in C_0^\infty(\overline{\Omega_j})$, $f_j := P_j u_j|_{\Omega_j}$, and u_j^0 be the extension of u_j by zero outside Ω_j. If we apply P_j in the distributional sense to u_j^0, the result differs from f_j^0 by a distribution supported by Γ [**6** (23.48.13.4)], [**1**]:

$$P_j u_j^0 = f_j^0 + (-1)^j \sum_{k=0}^{2m-1} \sum_{l=0}^{2m-1-k} P_{j,k+l+1}\, \partial_n^l u_j \otimes \partial_n^k \delta_\Gamma. \tag{2.8}$$

Multipole layers $v \otimes \partial_n^k \delta_\Gamma$ appear here, defined by

$$\langle v \otimes \partial_n^k \delta_\Gamma, \phi \rangle := \int_\Gamma v\, \partial_n'^k \phi\, do \quad \text{for} \quad \phi \in C_0^\infty(\mathbb{R}^n), \tag{2.9}$$

where do is the $(n-1)$-dimensional surface measure on Γ, and ∂_n' is the transpose of the differential operator ∂_n. Thus (2.8) is a distributional formulation of Green's second formula. By applying the relation $G_j P_j u_j^0 = u_j^0$ to (2.8), we obtain the representation formula:

$$u_j^0 = G_j f_j^0 + (-1)^j \sum_{k=0}^{2m-1} \sum_{l=0}^{2m-1-k} K_{jk}(P_{j,k+l+1}\, \partial_n^l u_j). \tag{2.10}$$

Here the multipole potential operators K_{jk} are defined by

$$K_{jk}\phi = G_j(\phi \otimes \partial_n^k \delta_\Gamma) = \int_\Gamma \partial_{n(y)}'^k G_j(\cdot, y)\phi(y)\, do(y) \quad \text{for} \quad \phi \in C^\infty(\Gamma), \tag{2.11}$$

where $G_j(x, y)$ is the kernel of G_j.

If we introduce the matrices

$$\begin{aligned} \mathcal{P}_j &:= (P_{j,k+l+1})_{k,l=0}^{2m-1} \quad \text{with} \quad P_{jk} := 0 \quad \text{for} \quad k > 2m, \\ \mathcal{K}_j &:= (\partial_n^i K_{jk}|_{\partial\Omega_j})_{i,k=0}^{2m-1}, \end{aligned} \tag{2.12}$$

then we find from (2.10) by taking Cauchy data, that

$$\gamma_j u_j = \gamma_j G_j f_j^0 + (-1)^j \mathcal{K}_j \mathcal{P}_j \gamma_j u_j. \tag{2.13}$$

The operator

$$C_j := (-1)^j \mathcal{K}_j \mathcal{P}_j \tag{2.14}$$

is the so-called "Calderón projector". Its properties are known [22]:

Lemma 2.1. *The operator C_j is a pseudodifferential operator $C_j = (C_j^{ik})_{i,k=0}^{2m-1}$ with orders* ord $C_j^{ik} = i - k$. *Thus it is a continuous operator from*

$$\mathcal{H}^s := \prod_{k=0}^{2m-1} H^{m-k-1/2+s}(\Gamma) \tag{2.15}$$

into itself for any $s \in \mathbb{R}$. If $P_1 = P_2$ then

$$C_1 + C_2 = 1. \tag{2.16}$$

Here $H^s(\Gamma)$ is the usual Sobolev space on Γ.

From (2.13) it follows immediately that C_j are projection operators: $C_j^2 = C_j$. For Ω_1 we can take closures in Sobolev spaces in (2.10), whereas for the exterior domain Ω_2 we have (2.10) at first only for functions u_2 with compact support. But we can use (2.10) for

$$\Omega_2^R := \Omega_2 \cap \{x \in \mathbb{R}^n \mid |x| \leqq R\} \quad (R \text{ large enough}).$$

Then (2.10) holds for Ω_j if and only if the terms coming from $\Omega_2 \backslash \Omega_2^R$ and from $\partial\Omega_2^R \backslash \Gamma = \{x \in \mathbb{R}^n \mid |x| = R\}$ tend to zero as $R \to \infty$. For $f_2 = 0$ this means

$$\lim_{R\to\infty} \sum_{k+l+1 \leqq 2m} \int_{|y|=R} \partial_{n(y)}^{\prime k} G_2(\cdot, y)(P_{2,k+l+1}\, \partial_n^l u_2)(y)\, do(y) \to 0 \text{ for all } x \in \Omega_2. \tag{2.17}$$

Analogously to the classical situation of the Helmholtz and Maxwell equations, we call this a "*radiation condition*".

We can now define the solution spaces for (2.1), (2.2):

Definition 2.2.

$$\begin{aligned} L_1^s &:= \{u_1 \in H^{m+s}(\Omega_1) \mid P_1 u_1 = 0 \text{ in } \Omega_1\} \\ L_2^s &:= \{u_2 \in H^{m+s}(\Omega_2) \mid P_2 u_2 = 0 \text{ in } \Omega_2 \text{ and } u_2 \text{ satisfies (2.17)}\}. \end{aligned}$$

Note that the ellipticity of P_2 implies $u_2 \in C^\infty(\Omega_2)$, so that the integral in (2.17) makes sense. If we use the well-known mapping properties of the potential operators K_{jk} ([**1**], [**6**]), we obtain

LEMMA 2.3. *Let $s \in \mathbb{R}$. For $v \in \mathcal{H}^s$ the following are equivalent:*

(*i*) $C_j v = v$.
(*ii*) $C_j g = v$ *for some* $g \in \mathcal{H}^s$.
(*iii*) $v = \gamma_j u_j$ *for some* $u_j \in L_j^s$.

In this case, u_j is given by the representation formula

$$u_j = K_j \mathcal{P}_j v \text{ in } \Omega_j \tag{2.18}$$

where K_j is the vector $(K_{j0}, \ldots, K_{j,2m-1})$.

Thus we can write problem (2.1)–(2.3) in the equivalent form

$$(1 - C_1)\gamma_1 u_1 = 0, \tag{2.19}$$

$$(1 - C_2)\gamma_2 u_2 = 0, \tag{2.20}$$

$$R_2\gamma_2 u_2 - R_1\gamma_1 u_1 = u_0. \tag{2.21}$$

This is a system of $6m$ equations on the boundary for the $4m$ unknowns $\gamma_1 u_1$, $\gamma_2 u_2$.

We emphasise here that up to now everything remains true if we consider u_1 and u_2 as vector-valued functions with N components and P_1 and P_2 as $(N \times N)$-systems of differential operators. The matrices R_j, $\mathcal{P}_j$, $\mathcal{K}_j$, C_j, etc., must then be block matrices of $(N \times N)$-blocks. System (2.19), (2.20), (2.21) is then a $(6mN \times 4mN)$-system.

From this system we can extract a quadratic subsystem by eliminating $R_2\gamma_2 u_2$ from (2.21) and multiplying (2.19), (2.20) by R_1 and R_2, respectively. We obtain

$$(1 - \tilde{C}_1)R_1\gamma_1 u_1 = 0, \tag{2.22}$$

$$(1 - \tilde{C}_2)R_1\gamma_1 u_1 = -(1 - \tilde{C}_2)u_0, \tag{2.23}$$

with

$$\tilde{C}_j := R_j C_j R_j^{-1} \quad (j = 1, 2). \tag{2.24}$$

Now we subtract (2.22) from (2.23) and obtain the quadratic system

$$Hv = -(1 - \tilde{C}_2)u_0 \quad \text{with} \quad H := \tilde{C}_1 - \tilde{C}_2 \tag{2.25}$$

for the unknown $v = R_1\gamma_1 u_1$.

We have the following equivalence theorem:

THEOREM 2.4. *Let $u_0 \in \mathcal{H}^s$ be given.*

(*i*) *If $u_j \in L_j^s (j = 1, 2)$ solve the transmission problem* (2.1), (2.2), (2.3) *then $v = R_1\gamma_1 u_1 \in \mathcal{H}^s$ solves the equation* (2.25).

(*ii*) *If $v \in \mathcal{H}^s$ solves* (2.25) *then with*

$$v_1 := \tilde{C}_1 v; \quad v_2 := \tilde{C}_2(v + u_0) \tag{2.26}$$

and

$$u_j := K_j \mathcal{P}_j R_j^{-1} v_j \text{ in } \Omega_j \quad (\text{see } (2.18)), \tag{2.27}$$

$u_j \in L_j^s$ solve the problem (2.1), (2.2), (2.3).

Proof. (i) This follows from the derivation of (2.25) above.

(ii) We use Lemma 2.3 which, from (2.26), gives that (for $j=1, 2$) $(1-\tilde{C}_j)v_j=0$, hence $v_j = R_j\gamma_j\mu_j$ for some $u_j \in L_j^s$. It remains to show (2.3):
From (2.26) and (2.25) it follows that

$$v_2 - v_1 = (\tilde{C}_2 - \tilde{C}_1)v + \tilde{C}_2 u_0 = -Hv + \tilde{C}_2 u_0 = (1-\tilde{C}_2+\tilde{C}_2)u_0 = u_0. \qquad \square$$

Now we formulate the assumptions which will imply the strong ellipticity of the operator H. They consist essentially of the strong ellipticity of the boundary value problems on Ω_1 and Ω_2 and of two other boundary value problems obtained from interchanging the domains Ω_1 and Ω_2.

We require the existence of a "first Green formula" for P_j and R_j: Let

$$R_j := \begin{pmatrix} B_j^0 \\ \vdots \\ B_j^{m-1} \\ Q_j^{m-1} \\ \vdots \\ Q_j^0 \end{pmatrix}, \quad \text{i.e.} \quad \left.\begin{matrix} B_j^{ik} = R_j^{ik} & \text{for } k=0,\dots,m-1, \\ Q_j^{ik} = R_j^{i,2m-k-1} & \text{for } k=0,\dots,m-1; \end{matrix}\right\} \tag{2.28}$$

and $B_j^i v := \sum_{k=0}^{m-1} B_j^{ik} v^k$ for $v \in C^\infty(\Gamma; \mathbb{C}^m)$, Q_j^i correspondingly.

Assumption 2.5. For $j=1, 2$, there exists a sesquilinear form Φ_j: $(u, v) \mapsto \Phi_j(u, v)$ on $C_0^\infty(\overline{\Omega_j}) \times C_0^\infty(\overline{\Omega_j})$ such that

$$\operatorname{Re} \int_{\Omega_j} \overline{u_j} \cdot P_j u_j \, dx = \operatorname{Re} \Phi_j(u_j, u_j) + (-1)^j \operatorname{Re} \int_\Gamma \sum_{i=0}^{m-1} \overline{B_j^i \gamma_j u_j} \cdot Q_j^i \gamma_j u_j \, do \tag{2.29}$$

for $u_j \in C_0^\infty(\overline{\Omega_j})$.

Assumption 2.6. (a) (Continuity). For every bounded subset $K \subset \mathbb{R}^n$ there exists $C > 0$ such that (for $j=1, 2$)

$$|\Phi_j(u, v)| \leqq C \, \|u\|_{H^m(\Omega_j)} \, \|v\|_{H^m(\Omega_j)}$$

for all $u, v \in C_0^\infty(\overline{\Omega_j} \cap K)$.

(b) (Gårding's inequality) For every bounded subset $K \subset \mathbb{R}^n$ there exist $\lambda > 0$, $c \in \mathbb{R}$, $\varepsilon > 0$ such that

$$\operatorname{Re} \Phi_j(u, u) \geqq \lambda \, \|u\|^2_{H^m(\Omega_j)} - c \, \|u\|^2_{H^{m-\varepsilon}(\Omega_j)} \tag{2.30}$$

for all $u \in C_0^\infty(\overline{\Omega_j} \cap K)$.

It is classical (see e.g. **[17]**) that these assumptions are satisfied for many strongly elliptic boundary value problems. For example, let P_j be of second order, i.e.

$$P_j = -\sum_{i,k=1}^n \partial_i a_j^{ik} \partial_k + \sum_{k=1}^n b_j^k \partial_k + c_j \tag{2.31}$$

with smooth coefficients a_j^{ik}, b_j^k, and c_j; $(\partial_k = \partial/\partial_{x_k})$.

Green's first formula (2.29) holds with

$$\Phi_j(u, v) = \int_{\Omega_j} \left(\sum_{i,k=1}^{n} \overline{\partial_i u} \,.\, a_j^{ik}\, \partial_k v + \sum_{k=1}^{n} \bar{u} \,.\, b_j^k\, \partial_k v + \bar{u} \,.\, c_j v \right) dx \tag{2.32}$$

and

$$\begin{aligned} B_j \gamma_j u_j &= u|_\Gamma, \\ Q_j \gamma_j u_j &= \partial_{\nu_j} u|_\Gamma, \end{aligned} \tag{2.33}$$

with the conormal derivative

$$\partial_{\nu_j} u = \sum_{i,k=1}^{n} n_i a_j^{ik}\, \partial_k u. \tag{2.34}$$

Here u may be an N-vector and a_j^{ik}, b_j^{ik}, and c_j $(N \times N)$-matrices. The representation (2.31) is not unique, and, contrary to the scalar case, in the vector-valued case the validity of Assumption 2.6(b) depends on the choice of this divergence representation. This is what happens for the case of electromagnetic problems, see Section 3.

The boundary integral in (2.29) corresponds to the natural duality on the "energy space" $\mathcal{H}^0$ with respect to the $L^2(\Gamma; \mathbb{C}^{2m})$ scalar product:

For $v, w \in C^\infty(\Gamma; \mathbb{C}^m)$ with

$$v = \begin{pmatrix} v_0 \\ \vdots \\ v_{2m-1} \end{pmatrix}, \quad w = \begin{pmatrix} w_0 \\ \vdots \\ w_{2m-1} \end{pmatrix}$$

let

$$(v, w)_{\mathcal{H}^0} := \int_\Gamma \sum_{k=0}^{2m-1} \overline{v_k} \,.\, w_{2m-1-k}\, do. \tag{2.35}$$

This then extends by continuity to $v, w \in \mathcal{H}^0$ (cf. (2.15)), and we have

$$(R_j v, R_j w)_{\mathcal{H}^0} = \int_\Gamma \left\{ \sum_{k=0}^{m-1} \overline{B_j^k v} \,.\, Q_j^k w + \sum_{k=0}^{m-1} \overline{Q_j^k v} \,.\, B_j^k w \right\} do,$$

in particular for $v = w$

$$(R_j v, R_j v)_{\mathcal{H}^0} = 2\,\mathrm{Re} \int_\Gamma \sum_{k=0}^{m-1} \overline{B_j^k v} \,.\, Q_j^k v\, do. \tag{2.36}$$

We need to consider two more boundary value problems defined by interchanging the interior and exterior domains. Thus we write

$$\left. \begin{aligned} P_1^\dagger &:= P_2 \text{ on } \Omega_1, & P_2^\dagger &:= P_1 \text{ on } \Omega_2, \\ R_1^\dagger &:= R_2, & R_2^\dagger &:= R_1. \end{aligned} \right\} \tag{2.37}$$

ASSUMPTION 2.7. Assumptions 2.5, 2.6 are satisfied if P_j is replaced by $P_j^\dagger$ and R_j by $R_j^\dagger$ for $j = 1, 2$.

Under these assumptions, we can infer the strong ellipticity of our boundary integral operator H, as follows:

THEOREM 2.8. *Let Assumptions* 2.5, 2,6, 2.7 *be satisfied. Then there exists a* *compact operator $C: \mathcal{H}^0 \to \mathcal{H}^0$ and a constant $\beta > 0$ such that*

$$\operatorname{Re}\,(v, (H + C)v)_{\mathcal{H}^0} \geqq \beta\, \|v\|^2_{\mathcal{H}^0} \quad \text{for all } v \in \mathcal{H}^0. \tag{2.38}$$

Here $\|v\|^2_{\mathcal{H}^0} = \sum_{k=0}^{2m-1} \|v_k\|^2_{H^{m-k-1/2}(\Gamma)}$.

Proof. We first consider the special case

$$P_1 = P_2, \quad R_1 = R_2. \tag{2.39}$$

In this case, one finds from (2.16), (2.24) that

$$\tilde{C}_1 + \tilde{C}_2 = 1. \tag{2.40}$$

It suffices to show (2.38) for $v \in C^\infty(\Gamma; \mathbb{C}^m) \subset \mathcal{H}^0$. Define u_j $(j = 1, 2)$ by the representation formula (2.27):

$$u_j = \chi K_j \mathcal{P}_j R_j^{-1} v \quad \text{in } \Omega_j, \tag{2.41}$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\chi \equiv 1$ on a neighbourhood of $\overline{\Omega_1}$. Then from (2.24) and (2.14) we see that, for the Cauchy data,

$$R_j \gamma_j u_j = (-1)^j \tilde{C}_j v \tag{2.42}$$

holds.

Thus, by the definition of H in (2.25),

$$Hv = -(R_1 \gamma_1 u_1 + R_2 \gamma_2 u_2). \tag{2.43}$$

By (2.40) we have

$$v = R_2 \gamma_2 u_2 - R_1 \gamma_1 u_1. \tag{2.44}$$

Thus the trace lemma gives

$$\|v\|^2_{\mathcal{H}^0} \leqq C(\|u_1\|^2_{H^m(\Omega_1)} + \|u_2\|^2_{H^m(\Omega_2)}). \tag{2.45}$$

By Gårding's inequality (2.30), the right-hand side can be further estimated by

$$\frac{C}{\lambda} \operatorname{Re}\,(\Phi_1(u_1, u_1) + \Phi_2(u_2, u_2)) + \frac{cC}{\lambda} (\|u_1\|^2_{H^{m-\varepsilon}(\Omega_1)} + \|u_2\|^2_{H^{m-\varepsilon}(\Omega_2)}). \tag{2.46}$$

Note that $\operatorname{supp} u_2 \subset K$ where $K = \operatorname{supp} \chi$ is bounded.

From (2.41) it follows that the $\|\cdot\|_{H^{m-\varepsilon}}$-terms in (2.46) can be estimated by $\|T_1 v\|_{\mathcal{H}^0}$ with a compact operator $T_1: \mathcal{H}^0 \to \mathcal{H}^0$. Such an estimate is also possible for $\|P_2 u_2\|_{H^t(\Omega_2)}$ for any $t \in \mathbb{R}$, because $P_2 u_2 = 0$ where $\chi \equiv 1$ or $\chi \equiv 0$ holds, thus $P_2 u_2 \in C_0^\infty(\mathbb{R}^n)$, and its support has a positive distance from Γ, so that the kernel of the operator $P_j \chi K_j \mathcal{P}_j R_j^{-1}$ defining it is smooth. Thus

$$\left| \int_{\Omega_2} \overline{u_2} \,.\, P_2 u_2 \, dx \right| \leqq \|T_2 v\|^2_{\mathcal{H}^0} \quad \text{for some compact } T_2: \mathcal{H}^0 \to \mathcal{H}^0.$$

Now we apply Green's first theorem (2.29) to (2.46) and use (2.36). With $P_1 u_1 = 0$, we obtain

$$\operatorname{Re} \Phi_j(u_j, u_j) = \operatorname{Re} \int_{\Omega_j} \overline{u_j} \,.\, P_j u_j \, dx - \frac{(-1)^j}{2} (R_j \gamma_j u_j, R_j \gamma_j u_j)_{\mathcal{H}^0},$$

hence

$$\|v\|^2_{\mathcal{H}^0} \leqq \|T_1 v\|^2_{\mathcal{H}^0} + \|T_2 v\|^2_{\mathcal{H}^0} + \frac{C}{2\lambda}\{(R_1\gamma_1 u_1, R_1\gamma_1 u_1)_{\mathcal{H}^0} - (R_2\gamma_2 u_2, R_2\gamma_2 u_2)_{\mathcal{H}^0}\}. \tag{2.47}$$

On the other hand, we find, from (2.43), (2.44), that

$$\begin{aligned}(v, Av)_{\mathcal{H}^0} &= (-R_1\gamma_1 u_1 + R_2\gamma_2 u_2, -R_1\gamma_1 u_1 - R_2\gamma_2 u_2)_{\mathcal{H}^0}\\ &= (R_1\gamma_1 u_1, R_1\gamma_1 u_1)_{\mathcal{H}^0} - (R_2\gamma_2 u_2, R_2\gamma_2 u_2)_{\mathcal{H}^0}\\ &\quad + (R_1\gamma_1 u_1, R_2\gamma_2 u_2)_{\mathcal{H}^0} - (R_2\gamma_2 u_2, R_1\gamma_1 u_1)_{\mathcal{H}^0}\\ &= (R_1\gamma_1 u_1, R_1\gamma_1 u_1)_{\mathcal{H}^0} - (R_2\gamma_2 u_2, R_2\gamma_2 u_2)_{\mathcal{H}^0}\\ &\quad + 2i \operatorname{Im}(R_1\gamma_1 u_1, R_2\gamma_2 u_2)_{\mathcal{H}^0}.\end{aligned}$$

By taking real parts, we conclude from (2.47) that

$$\|v\|^2_{\mathcal{H}^0} \leqq \|T_1 v\|^2_{\mathcal{H}^0} + \|T_2 v\|^2_{\mathcal{H}^0} + \frac{C}{2\lambda}\operatorname{Re}(v, Av)_{\mathcal{H}^0}.$$

After subsuming all compact parts into a single one, we arrive at (2.38).

Now we abandon hypothesis (2.39). By what we have shown so far, we know that Gårding's inequality (2.38) holds in particular for the case where P_2 and R_2 are replaced by $P_2^\dagger$ and $R_2^\dagger$, respectively, because by (2.37) hypothesis (2.39) is then satisfied. We denote the corresponding Calderón projector by $\tilde{C}_2^\dagger$, and the corresponding boundary integral operator by

$$H_1^\dagger := \tilde{C}_1 - \tilde{C}_2^\dagger.$$

Similarly, if we replace P_1 and R_1 by $P_1^\dagger$ and $R_1^\dagger$, then (2.39) is satisfied and therefore Gårding's inequality holds. We denote the corresponding Calderón projector by $\tilde{C}_1^\dagger$ and the boundary integral operator by

$$H_2^\dagger := \tilde{C}_1^\dagger - \tilde{C}_2.$$

Now from (2.40), it follows that

$$\tilde{C}_1 + \tilde{C}_2^\dagger = 1 = \tilde{C}_1^\dagger + \tilde{C}_2,$$

hence

$$H_1^\dagger = 2\tilde{C}_1 - 1; \quad H_2^\dagger = 1 - 2\tilde{C}_2$$

and finally

$$H = \tilde{C}_1 - \tilde{C}_2 = \tfrac{1}{2}(H_1^\dagger + H_2^\dagger). \tag{2.48}$$

Therefore, the Gårding inequality (2.38) for H follows by adding the two Gårding inequalities for $H_1^\dagger$ and $H_2^\dagger$. □

3. A coercive bilinear form for electromagnetic problems

The time-harmonic scattering of electromagnetic fields by a penetrable body Ω_1 in the case of isotropic homogeneous materials is described by Maxwell's equations (see [**20**]):

$$\operatorname{curl}\vec{E} = i\omega\mu\vec{H}; \qquad \operatorname{curl}\vec{H} = -i\omega\varepsilon\vec{E} \quad \text{in} \quad \Omega_1 \cup \Omega_2, \tag{3.1}$$

with the transmission conditions

$$[\vec{n}\times\vec{E}]_\Gamma=0;\quad [\vec{n}\times\vec{H}]_\Gamma=0\quad \text{on } \Gamma \tag{3.2}$$

and the radiation condition

$$\omega\mu\frac{x}{|x|}\times\vec{H}_{sc}+k\vec{E}_{sc}=o(|x|^{-1});\quad \vec{E}_{sc}=O(|x|^{-1})\quad \text{as}\quad |x|\to\infty. \tag{3.3}$$

Here $[\vec{v}]_\Gamma:=\vec{v}\,|_{\overline{\Omega_2}}-\vec{v}|_{\overline{\Omega_1}}$ denotes the jump of $\vec{v}$ across Γ.

We assume that the coefficient functions ε, μ, and k are constant on Ω_1 and on Ω_2, and ω is the constant frequency:

$$\varepsilon=\varepsilon_j,\quad \mu=\mu_j,\quad k=k_j\quad \text{on}\quad \Omega_j\ (j=1,2);\quad k^2=\omega^2\varepsilon\mu.$$

We further assume ([**20**]) that

$$\operatorname{Re}\varepsilon_j>0,\quad \operatorname{Im}\varepsilon_j\geqq 0;\quad \arg k_j\in[0,\pi);\quad \arg\omega\in[0,\pi);\quad \mu_j>0. \tag{3.4}$$

The total fields $\vec{E}$ are decomposed into

$$\vec{E}=\vec{E}_{in}+\vec{E}_{sc};\quad \vec{H}=\vec{H}_{in}+\vec{H}_{sc},$$

where the incoming fields $\vec{E}_{in}$ and $\vec{H}_{in}$ are supposed to satisfy (3.1), and the scattered fields $\vec{E}_{sc}$ and $\vec{H}_{sc}$ appear in the radiation condition (3.3). The unique solvability of (3.1)–(3.3) is shown in [**16**], [**20**].

If we define

$$\vec{u}=\vec{u}_1=\vec{E}\quad \text{in } \Omega_1;\quad \vec{u}=\vec{u}_2=\vec{E}_{sc}\quad \text{in } \Omega_2;\quad \vec{u}_0=\vec{E}_{in}, \tag{3.5}$$

then the components of $\vec{u}$ satisfy the Helmholtz equation:

$$(\Delta+k_1^2)\vec{u}_1=0\quad \text{in } \Omega_1;\quad (\Delta+k_2^2)\vec{u}_2=0\quad \text{in } \Omega_2. \tag{3.6}$$

We consider the transmission conditions (compare [**16**])

$$\left.\begin{aligned} \vec{u}_{1T}-\vec{u}_{2T}&=\vec{u}_{0T};\\ \lambda_1\operatorname{div}\vec{u}_1-\lambda_2\operatorname{div}\vec{u}_2&=\lambda_2\operatorname{div}\vec{u}_0;\\ \varepsilon_1\vec{n}\,.\,\vec{u}_1-\varepsilon_2\vec{n}\,.\,\vec{u}_2&=\varepsilon_2\vec{n}\,.\,\vec{u}_0;\\ \frac{1}{\mu_1}\vec{n}\times\operatorname{curl}\vec{u}_1-\frac{1}{\mu_2}\vec{n}\times\operatorname{curl}\vec{u}_2&=\frac{1}{\mu_2}\vec{n}\times\operatorname{curl}\vec{u}_0, \end{aligned}\right\} \tag{3.7}$$

where $\vec{v}_T:=-\vec{n}\times(\vec{n}\times\vec{v})$ denote the tangential components of $\vec{v}$, and the radiation condition

$$\frac{x}{|x|}\times\operatorname{curl}\vec{u}_2-\frac{x}{|x|}\operatorname{div}\vec{u}_2+ik_2\vec{u}_2=o(|x|^{-1})\quad \text{as}\quad |x|\to\infty. \tag{3.8}$$

The coefficients $\lambda_j\neq 0$ in (3.7) are specified later.

Let us now study the equivalence of the transmission problems (3.1)–(3.3) and (3.6)–(3.8). Because we only need standard applications of Green's formulae, we do not specify the precise smoothness requirements. Even weak solutions in $H^1_{loc}(\overline{\Omega_j})$ are allowed.

We first show that (3.1)–(3.3) imply (3.6)–(3.8), for any choice of λ_1, λ_2:

By the definition (3.5), it is clear that Maxwell's equations (3.1) imply the Helmholtz equations (3.6), and $\operatorname{div}\vec{E}=0$ shows that the radiation condition (3.8) follows from (3.3). Also the first, second, and fourth of the transmission conditions (3.7) are satisfied. In order to show the third condition in (3.7), we choose a test function $\varphi\in C_0^\infty(\mathbb{R}^3)$ and obtain

$$\begin{aligned} 0&=\int_\Gamma \operatorname{grad}\varphi\,.\,[\vec{n}\times\vec{H}]_\Gamma\,do=-\int_{\Omega_1\cup\Omega_2}\operatorname{grad}\varphi\,.\,\operatorname{curl}\vec{H}\,dx\\ &=i\omega\int_{\Omega_1\cup\Omega_2}\operatorname{grad}\varphi\,.\,\varepsilon\vec{E}\,dx=-i\omega\int_\Gamma\varphi[\varepsilon\vec{n}\,.\,\vec{E}]_\Gamma\,do.\end{aligned}\tag{3.9}$$

From this we find $[\varepsilon\vec{n}\,.\,\vec{E}]_\Gamma=0$, which is the third condition in (3.7).

Conversely, assume that (3.6)–(3.8) are satisfied. Furthermore, assume that $\vec{u}_0$ satisfies $(\Delta+k_2^2)\vec{u}_0=0$ and $\operatorname{div}\vec{u}_0=0$, $\vec{E}$ is defined by (3.5), and $\vec{H}$ is defined by $\vec{H}=(1/i\omega\mu)\operatorname{curl}\vec{E}$. Then (3.1)–(3.3) will be satisfied if and only if $\operatorname{div}\vec{u}_1\equiv0$ and $\operatorname{div}\vec{u}_2\equiv0$. Therefore we define

$$\rho:=\rho_1:=k_1^{-2}\operatorname{div}\vec{u}_1\quad\text{in }\Omega_1;\quad \rho:=\rho_2:=k_2^{-2}\operatorname{div}\vec{u}_2\,(=k_2^{-2}\operatorname{div}(\vec{u}_0+\vec{u}_2))\quad\text{in }\Omega_2.$$

Then ρ satisfies

$$(\Delta+k^2)\rho=0\quad\text{in }\Omega_1\cup\Omega_2\tag{3.10}$$

and

$$\lambda_1k_1^2\rho_1=\lambda_2k_2^2\rho_2\quad\text{on }\Gamma.\tag{3.11}$$

Furthermore, from the radiation condition (3.8), it follows (**[14]**, **[25]**) that u_2 can be represented in Ω_2 by the Stratton–Chu representation formula (see Lemma 4.3, below). This implies in particular that ρ satisfies a Sommerfeld type radiation condition

$$\frac{x}{|x|}\cdot\operatorname{grad}\rho-ik_2\rho=o(|x|^{-1});\quad \rho=O(|x|^{-1})\quad\text{as}\quad|x|\to\infty.\tag{3.12}$$

In addition, a second transmission condition for ρ holds:

Define $\vec{E}_0:=\vec{E}+\operatorname{grad}\rho$ in $\Omega_1\cup\Omega_2$. Then

$$\operatorname{div}\vec{E}_0=\operatorname{div}\vec{E}+\Delta\rho=\operatorname{div}\vec{u}-k^2\rho=0\quad\text{in}\quad\Omega_1\cup\Omega_2.$$

Hence

$$\operatorname{curl}\vec{H}=\frac{1}{i\omega\mu}\operatorname{curl}\operatorname{curl}\vec{E}_0=-i\omega\varepsilon\vec{E}_0.$$

Thus the pair $(\vec{E}_0,\vec{H})$ satisfies Maxwell's equations (3.1) and the transmission condition $[\vec{n}\times\vec{H}]_\Gamma=0$. We conclude as above (3.9) that $[\varepsilon\vec{n}\,.\,\vec{E}_0]_\Gamma=0$. Subtracting $[\varepsilon\vec{n}\,.\,\vec{E}]_\Gamma=0$, we find

$$\varepsilon_1\,\partial_n\rho_1=\varepsilon_2\,\partial_n\rho_2\quad\text{on }\Gamma.\tag{3.13}$$

Thus we have reduced the question of equivalence of the two transmission problems to the question of unique solvability of the scalar transmission problem (3.10)–(3.13). Sufficient conditions for this uniqueness are well-known. For example, from [**4**, Proposition 4.7] it follows that either one of the following two conditions implies $\rho \equiv 0$:

$$k_2 > 0 \quad \text{and} \quad \operatorname{Im} \lambda_1 \overline{\lambda_2 \varepsilon_1} \varepsilon_2 k_1^2 \geqq 0 \quad \text{and} \quad \operatorname{Im} \lambda_1 \overline{\lambda_1 \varepsilon_1} \varepsilon_2 \leqq 0; \tag{3.14}$$

$$\left.\begin{array}{l} \operatorname{Im} k_2 > 0 \quad \text{or} \quad k_2 = 0, \quad \text{and if there exist four numbers } \alpha, \beta, \gamma, \delta \geqq 0 \\ \text{with} \quad -\alpha \lambda_1 \overline{\varepsilon_1} - \beta \lambda_2 \overline{\varepsilon_2} + \gamma \lambda_1 \overline{\varepsilon_1} k_1^2 + \delta \lambda_2 \overline{\varepsilon_2} k_2^2 = 0, \\ \text{then at least one of the numbers } \alpha, \beta, \gamma, \delta \text{ has to be zero.} \end{array}\right\} \tag{3.15}$$

Notable special cases of these conditions are:

(i) If $\lambda_1 = \varepsilon_1$ and $\lambda_2 = \varepsilon_2$, then $\rho \equiv 0$ follows.
(ii) If $\lambda_1 = 1/\mu_1 \overline{\varepsilon_1}$ and $\lambda_2 = 1/\mu_2 \overline{\varepsilon_2}$, then $\rho \equiv 0$ follows.
(iii) If all coefficients ε, λ, and k_2 are real, then $\rho \equiv 0$ follows.
(iv) The periodic eddy current problem:
Here $\varepsilon_1 = i\sigma/\omega$, $\omega > 0$, and $\sigma > 0$ is the electric conductivity in Ω_1. Also $\varepsilon_2 > 0$, hence

$$k_1^2 = i\omega\mu_1\sigma; \quad k_2^2 > 0.$$

Therefore, (3.14) reduces to the conditions

$$\operatorname{Im} \frac{\lambda_2}{\lambda_1} \leqq 0 \quad \text{and} \quad \operatorname{Re} \frac{\lambda_2}{\lambda_1} \geqq 0. \tag{3.16}$$

Note that $\operatorname{Im}(k_1^2/k_2^2) > 0$, so that in this case the natural choice

$$\lambda = k^{-2} \tag{3.17}$$

does not necessarily imply $\rho \equiv 0$. The choices (i) or (ii) above, however, will also work in this case.

(v) The choice (3.17) always leads to a solution of the transmission problem (3.1)–(3.3). Namely, in this case define $\vec{E}_0$ as above by $\vec{E}_0 = \vec{E} + \operatorname{grad} \rho$, then the pair $(\vec{E}_0, \vec{H})$ satisfies Maxwell's equations (3.1) as well as the transmission conditions (3.2). However, as seen in (iv) above, the solution of the problem (3.6)–(3.8) might then be non-unique.

Besides the "physical" transmission problem, for mathematical simplicity we also consider the corresponding problem where all the coefficients λ, ε, and μ are equal to unity, i.e. the transmission conditions

$$\left.\begin{array}{r} \vec{u}_{1T} - \vec{u}_{2T} = \vec{u}_{0T}; \\ \operatorname{div} \vec{u}_1 - \operatorname{div} \vec{u}_2 = \operatorname{div} \vec{u}_0; \\ \vec{n} \cdot \vec{u}_1 - \vec{n} \cdot \vec{u}_2 = \vec{n} \cdot \vec{u}_0; \\ \vec{n} \times \operatorname{curl} \vec{u}_1 - \vec{n} \times \operatorname{curl} \vec{u}_2 = \vec{n} \times \operatorname{curl} \vec{u}_0. \end{array}\right\} \tag{3.18}$$

Now we wish to apply the theory developed in Section 2 to the present case. We have the differential operators

$$P_j = -(\Delta + k_j^2) = \operatorname{curl} \operatorname{curl} - \operatorname{grad} \operatorname{div} - k_j^2. \tag{3.19}$$

This representation leads to the conormal derivatives (cf. (2.31), (2.34))

$$\partial_\nu \vec{u} := \partial_{\nu_1}\vec{u} = \partial_{\nu_2}\vec{u} = -\vec{n} \times \operatorname{curl} \vec{u} + \vec{n} \,.\, \operatorname{div} \vec{u} \tag{3.20}$$

and to the well-known [14] Green formula (cf. (2.29), (2.32))

$$-\int_{\Omega_j} \overline{\vec{u}_j} \,.\, (\Delta + k_j^2)\vec{w}_j \, dx = \int_{\Omega_j} (\operatorname{curl} \overline{\vec{u}_j} \,.\, \operatorname{curl} \vec{w}_j + \operatorname{div} \overline{\vec{u}_j} \operatorname{div} \vec{w}_j - k_j^2 \overline{\vec{u}_j} \,.\, \vec{w}_j) \, dx$$
$$+ (-1)^j \int_\Gamma \overline{\vec{u}_j} \,.\, (-\vec{n} \times \operatorname{curl} \vec{w}_j + \vec{n} \operatorname{div} \vec{w}_j) \, do \tag{3.21}$$

for $\vec{u}_j, \vec{w}_j \in C_0^\infty(\overline{\Omega_j}; \mathbb{C}^3)$.

So we have the sesquilinear forms

$$\Phi_j(\vec{u}, \vec{w}) = \int_{\Omega_j} (\operatorname{curl} \overline{\vec{u}} \,.\, \operatorname{curl} \vec{w} + \operatorname{div} \overline{\vec{u}} \operatorname{div} \vec{w} - k_j^2 \overline{\vec{u}} \,.\, \vec{w}) \, dx \tag{3.22}$$

and the boundary operators

$$R\vec{u} := R_1\vec{u} = R_2\vec{u} = \begin{pmatrix} \vec{u} \\ \partial_\nu \vec{u} \end{pmatrix}\Bigg|_\Gamma; \quad \begin{aligned} B\vec{u} &:= B_1\vec{u} = B_2\vec{u} = \vec{u}|_\Gamma; \\ Q\vec{u} &:= Q_1\vec{u} = Q_2\vec{u} = \partial_\nu \vec{u}|_\Gamma. \end{aligned} \tag{3.23}$$

By separating tangential and normal components and defining (on Γ) the Cauchy data

$$\begin{aligned} v &:= \vec{n} \,.\, \vec{u}; \quad \vec{v} := \vec{u}_T := \vec{u} - \vec{n}(\vec{n} \,.\, \vec{u}) = -\vec{n} \times (\vec{n} \times \vec{u}); \\ \psi &:= \operatorname{div} \vec{u}; \quad \vec{\psi} := -\vec{n} \times \operatorname{curl} \vec{u}, \end{aligned} \tag{3.24}$$

Green's first formula (3.21) reads

$$\int_{\Omega_j} \overline{\vec{u}^1} \,.\, P_j \vec{u}^2 \, dx = \Phi_j(\vec{u}^1, \vec{u}^2) + (-1)^j \int_\Gamma (\overline{v^1} \psi^2 + \overline{\vec{v}^1} \,.\, \vec{\psi}^2) \, do. \tag{3.25}$$

The corresponding second Green's formula, obtained by antisymmetrising (3.25), and representation formula ("Stratton–Chu formula", see Section 4), obtained by inserting a fundamental solution in the latter, are also well known. Hence one can derive boundary integral equations by the method described in Section 2. Assumptions 2.5 and 2.6(a) are satisfied. Assumption 2.6(b) does not hold, however, since for harmonic vector fields $\vec{u}$ we have

$$\Phi_j(\vec{u}, \vec{u}) = -\int_{\Omega_j} k_j^2 \, |\vec{u}|^2 \, dx,$$

which cannot be estimated from below by $\|\vec{u}\|^2_{H^1(\Omega_j;\mathbb{C}^3)}$. This well-known fact ([**8**], [**25**]) implies here that the boundary integral operator H is not strongly elliptic. There is, however, a strongly elliptic boundary integral operator for the transmission problem (3.6), (3.18). This gives, as the solution of the corresponding equation (2.25), not directly the set of Cauchy data as defined by (3.18) or (3.24) but a modified set. This modified set of Cauchy data is *entirely equivalent* to the original one in the sense that each set can be computed from the other one by application of tangential differential operators. We propose the following modification:

$$\left.\begin{aligned}\psi' &:= \psi - \operatorname{div}\vec{v} = \operatorname{div}\vec{u} - \operatorname{div}_T \vec{u},\\ \vec{\psi}' &:= \vec{\psi} + \operatorname{grad}_T v = -\vec{n}\times\operatorname{curl}\vec{u} - \vec{n}\times(\vec{n}\times(\operatorname{grad}(\vec{n}\,.\,\vec{u}))).\end{aligned}\right\} \tag{3.26}$$

Thus the "electric" boundary operators are modified by

$$\begin{pmatrix}\vec{v}\\ \psi'\end{pmatrix} = \begin{pmatrix}\vec{v}\\ \psi - \operatorname{div}_T \vec{v}\end{pmatrix} \Leftrightarrow \begin{pmatrix}\vec{v}\\ \psi\end{pmatrix} = \begin{pmatrix}\vec{v}\\ \psi' + \operatorname{div}_T \vec{v}\end{pmatrix} \tag{3.27}$$

and the "magnetic" boundary operators by

$$\begin{pmatrix}v\\ \vec{\psi}'\end{pmatrix} = \begin{pmatrix}v\\ \vec{\psi} + \operatorname{grad}_T v\end{pmatrix} \Leftrightarrow \begin{pmatrix}v\\ \vec{\psi}\end{pmatrix} = \begin{pmatrix}v\\ \vec{\psi}' - \operatorname{grad}_T v\end{pmatrix}. \tag{3.28}$$

This modification of the Cauchy data was suggested by corresponding modifications of boundary integral operators in the case of boundary value problems in **[18]**, which led to strongly elliptic operators.

We shall now show that with these modified boundary operators the corresponding bilinear forms Φ_j' which are now defined by Green's first formula, satisfy Gårding's inequality, i.e. Assumption 2.6(b). Then all hypotheses of Theorem 2.8 are satisfied and this theorem proves the strong ellipticity of the corresponding boundary integral operator, which corresponds to the transmission problem (3.6), (3.18), (3.8).

We shall come back to the "physical" transmission conditions (3.7) in Section 5.

THEOREM 3.1. *Define the sesquilinear form* Φ_j' $(j = 1, 2)$ *by*

$$\int_{\Omega_j} \overline{\vec{u}^1}\,.\,P_j\vec{u}^2\,dx = \Phi_j'(\vec{u}^1, \vec{u}^2) + (-1)^j \int_\Gamma (\overline{v^1}\psi'^2 + \overline{\vec{v}^1}\,.\,\vec{\psi}'^2)\,do, \tag{3.29}$$

where P_j are defined by (3.19) *and v, ψ', $\vec{v}$, $\vec{\psi}'$ by* (3.26). *Then Φ_j' is coercive over $H^1(\Omega_j; \mathbb{C}^3)$, i.e. there exist $\lambda > 0$, $C \in \mathbb{R}$ such that*

$$\operatorname{Re}\Phi_j'(\vec{u}, \vec{u}) \geqq \lambda\,\|\vec{u}\|^2_{H^1(\Omega_j;\mathbb{C}^3)} - C\,\|\vec{u}\|^2_{L^2(\Omega_j;\mathbb{C}^3)} \tag{3.30}$$

for all $\vec{u} \in C_0^\infty(\overline{\Omega_j}; \mathbb{C}^3)$.

Proof. There holds for $\vec{u} \in C_0^\infty(\overline{\Omega_j}; \mathbb{C}^3)$ on Γ:

$$\operatorname{div}\vec{u} - \operatorname{div}_T\vec{u} = \operatorname{div}(\vec{u} - \vec{u}_T) = \operatorname{div}((\vec{n}\,.\,\vec{u})\vec{n}) = \partial_n(\vec{n}\,.\,\vec{u}) + (\vec{n}\,.\,\vec{u})\operatorname{div}\vec{n};$$

$$\operatorname{grad}(\vec{n}\,.\,\vec{u}) - \vec{n}\times\operatorname{curl}\vec{n} = \partial_n\vec{u} + (\vec{u}\,.\,\operatorname{grad})\vec{n} + \vec{u}\times\operatorname{curl}\vec{n}.$$

This gives

$$\begin{aligned}\bar{v}\psi' + \bar{\vec{v}}\,.\,\vec{\psi}' &= (\operatorname{div}\vec{u} - \operatorname{div}_T\vec{u})(\vec{n}\,.\,\bar{\vec{u}}) + (\operatorname{grad}(\vec{n}\,.\,\vec{u}) - \vec{n}\times\operatorname{curl}\vec{u})\,.\,\bar{\vec{u}}_T\\ &= (\partial_n(\vec{u}\,.\,\vec{u}))(\vec{n}\,.\,\bar{\vec{u}}) + \operatorname{div}\vec{n}(\vec{n}\,.\,\vec{u})(\vec{n}\,.\,\bar{\vec{u}}) + (\partial_n\vec{u})\bar{\vec{u}}_T\\ &\quad + \bar{\vec{u}}_T\,.\,(\vec{u}\,.\,\operatorname{grad})\vec{n} + \bar{\vec{u}}_T\,.\,(\vec{u}\times\operatorname{curl}\vec{n})\\ &= \partial_n\vec{u}\,.\,(\vec{n}(\vec{n}\,.\,\bar{\vec{u}})) + \vec{u}\,.\,\partial_n\vec{n}(\vec{n}\,.\,\bar{\vec{u}}) + (\partial_n\vec{u})\bar{\vec{u}}_T\\ &\quad + \operatorname{div}\vec{n}\,|\vec{n}\,.\,\vec{u}|^2 + \bar{\vec{u}}_T\,.\,(\vec{u}\,.\,\operatorname{grad})\vec{n} + \bar{\vec{u}}_T\,.\,(\vec{n}\times\operatorname{curl}\vec{n})\\ &= (\partial_n\vec{u})\,.\,\bar{\vec{u}} + b(\vec{u})\end{aligned} \tag{3.31}$$

with

$$b(\vec{u}) = (\vec{n}\,.\,\bar{\vec{u}})\vec{u}\,.\,\partial_n\vec{n} + |\vec{n}\,.\,\vec{u}|^2\operatorname{div}\vec{n} + \bar{\vec{u}}_T\,.\,((\vec{u}\,.\,\operatorname{grad})\vec{n} + \vec{u}\times\operatorname{curl}\vec{n}).$$

In particular (this requires $\Gamma \in C^2$)

$$|b(\vec{u})(x)| \leqq C\,|\vec{u}(x)|^2 \quad \text{for} \quad x \in \Gamma,$$

hence

$$\left|\int_\Gamma b(\vec{u})\,do\right| \leqq C\,\|\vec{u}|_\Gamma\|^2_{L^2(\Gamma)} \leqq C_\varepsilon\,\|\vec{u}\|^2_{H^{\frac{1}{2}+\varepsilon}(\Omega_j)} \quad (\varepsilon>0). \tag{3.32}$$

From definition (3.29) and (3.31), with Green's first formula for the scalar potential equation, now follows:

$$\begin{aligned}\Phi_j'(\vec{u},\vec{u}) &= -\int_{\Omega_j}\bar{\vec{u}}\,.\,\Delta\vec{u}\,dx - (-1)^j\int_\Gamma \bar{\vec{u}}\,.\,\partial_n\vec{u}\,do\\ &\quad -\int_{\Omega_j}k_j^2\,|\vec{u}|^2\,dx - (-1)^j\int_\Gamma b(\vec{u})\,do\\ &= \|\vec{u}\|^2_{H^1(\Omega_j;\mathbb{C}^3)} - \|\vec{u}\|^2_{L^2(\Omega_j;\mathbb{C}^3)}\\ &\quad - k_j^2\,\|\vec{u}\|^2_{L^2(\Omega_j;\mathbb{C}^3)} - (-1)^j\int_\Gamma b(\vec{u})\,do.\end{aligned}$$

Together with (3.22) for some $\varepsilon \in (0, \frac{1}{2})$, this gives (3.30). □

4. The integral operators and their symbols

In this section we derive a boundary integral equation procedure to solve the transmission problem (3.6), (3.18), (3.8). First we give some additional notation. For $s \in \mathbb{R}$ we denote by $\mathbb{H}^s(\Omega_j) = H^s(\Omega_j;\mathbb{C}^3)$, respectively $\mathbb{H}^s(\Gamma) = H^s(\Gamma;\mathbb{C}^3)$, the Sobolev spaces formed by vector fields $\vec{u}$ with components which belong to $H^s(\Omega_j)$, respectively $H^s(\Gamma)$. As indicated by (3.24) we can decompose $\mathbb{H}^s(\Gamma)$ into two subspaces generated by the tangential fields to Γ and the normal fields to Γ,

$$\mathbb{H}^s(\Gamma) = TH^s(\Gamma) \oplus NH^s(\Gamma)$$

with

$$TH^s(\Gamma) = \{\vec{u} \in \mathbb{H}^s(\Gamma) \mid (\vec{n}\,.\,\vec{u}) = 0\}, \quad NH^s(\Gamma) = \{\vec{u} \in \mathbb{H}^s(\Gamma) \mid \vec{u} = \vec{n}v, \quad v \in H^s(\Gamma)\}.$$

The most general case where (3.6), (3.18), (3.8) can be converted into a variational problem (see Theorem 3.1) is when

$$(\vec{u}_{0T}, \operatorname{div}\vec{u}_0, \vec{n}\times\operatorname{curl}\vec{u}_0, \vec{n}\,.\,\vec{u}_0) \in TH^{\frac{1}{2}}(\Gamma) + H^{-\frac{1}{2}}(\Gamma)\times TH^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma) =: \mathscr{H}^0.$$

Then we look for $\vec{u} \in L_j$ $(j = 1, 2)$ where

$$\left.\begin{aligned} L_1 &:= \{\vec{u} \in \mathbb{H}^1(\Omega_1) \mid (\Delta + k_1^2)\vec{u} = 0 \text{ in } \Omega_1\},\\ L_2 &:= \{\vec{u} \in \mathbb{H}^1(\Omega_2) \mid (\Delta + k_2^2)\vec{u} = 0 \text{ in } \Omega_2,\ \vec{u} \text{ satisfies (3.8)}\}.\end{aligned}\right\} \tag{4.1}$$

According to (3.24), (3.25), the Cauchy data for (3.6), (3.18), (3.8) are defined as follows:

DEFINITION 4.1. Let $\vec{u} \in L_j$, $j = 1, 2$. Then the Cauchy data $(\vec{v}, \psi, \vec{\psi}, v) \in \mathscr{H}^0$ of

$\vec{u}$ are defined via (3.25) by the traces

$$\vec{v} := -\vec{n} \times (\vec{n} \times \vec{u})|_\Gamma, \quad \psi := \operatorname{div} \vec{u}|_\Gamma, \quad \vec{\psi} := -\vec{n} \times \operatorname{curl} \vec{u}|_\Gamma, \quad v := \vec{n} \,.\, \vec{u}|_\Gamma.$$

Before we give our solution procedure, let us briefly recall the idea of layer potentials by introducing the fundamental solution

$$\Phi_j(x, y) = \frac{e^{ik_j|x-y|}}{4\pi|x-y|} \tag{4.2}$$

of $(\Delta + k_j^2)\vec{u} = 0$ in $\Omega_j; j = 1, 2$.

DEFINITION 4.2. Let $\vec{u} \in C^\infty(\Gamma; \mathbb{C}^3)$. Then for any complex number k_j, $(0 \leqq \arg k_j < \pi)$ and for $x \in \Omega_j$ we define

$$\left.\begin{aligned} V_{\Omega_j}\vec{u}(x) &:= -2\int_\Gamma \Phi_j(x, y)\vec{u}(y)\, do(y), \\ K_{\Omega_j}\vec{u}(x) &:= 2 \operatorname{curl}_x \int_\Gamma \Phi_j(x, y)\vec{u}(y)\, do(y). \end{aligned}\right\} \tag{4.3}$$

The same definition of the single and double layer potential is valid for arbitrary distributions $\vec{u}$ on Γ, since for $x \neq \Gamma$ the above kernel Φ_j is a C^∞-function on Γ.

With these potentials there holds the Stratton–Chu representation formula for the solution of the homogeneous Helmholtz equation in Ω_j (for classical solutions see **[14]**, for weak solutions see **[25]**).

LEMMA 4.3. *For $\vec{u} \in L_j$ with Cauchy data $(\vec{v}, \psi, \vec{\psi}, v) \in \mathscr{H}^0$ and for $x \in \Omega_j$, $j = 1, 2$, there holds*

$$\vec{u}(x) = \frac{(-1)^j}{2}(-\operatorname{curl} V_{\Omega_j}(\vec{n} \times \vec{v}) + V_{\Omega_j}(\vec{n}\psi) + V_{\Omega_j}\vec{\psi} + \operatorname{grad} V_{\Omega_j}v)(x). \tag{4.4}$$

In order to formulate the boundary values (jump relations) for the potential (4.4) we define the following boundary integral operators:

DEFINITION 4.4. Let $\vec{u}$ be a C^∞ vector field on Γ. Then for $x \in \Gamma$ and $x_j \in \Omega_j$

$$\left.\begin{aligned} V_j\vec{u}(x) &:= -2\int_\Gamma \Phi_j(x, y)\vec{u}(y)\, do(y), \\ \vec{K}_j\vec{u}(x) &:= 2 \operatorname{curl}_T \int_\Gamma \Phi_j(x, y)(\vec{n} \times \vec{u})(y)\, do(y), \\ D_j\vec{u}(x) &:= \lim_{x_j \to x} (\vec{n} \times \operatorname{curl}\operatorname{curl} V_{\Omega_j}(\vec{n} \times \vec{u}))(x_j), \end{aligned}\right\} \tag{4.5}$$

and, correspondingly, for $u \in C^\infty(\Gamma)$

$$\left.\begin{aligned} V_j u(x) &:= -2\int_\Gamma y(y)\Phi_j(x, y)\, do(y), \\ K_j u(x) &:= -2\int_\Gamma u(y)\, \partial_{n(y)}\Phi_j(x, y)\, do(y), \\ K_j' u(x) &:= -2\int_\Gamma u(y)\, \partial_{n(x)}\Phi_j(x, y)\, do(x). \end{aligned}\right\} \tag{4.6}$$

Using the well-known jump relations for smooth layers ([**10**], [**2**]) and approximating the Cauchy data in $\mathscr{H}^0$ by smooth functions, we find for the traces of the potential (4.4) a system of second kind Fredholm integral equations on the boundary Γ:

$$\begin{aligned}2(-1)^j(-\vec{n}\times(\vec{n}\times\vec{u}))|_\Gamma &= ((-1)^j-\vec{K}_j)\vec{v}-\vec{n}\times(\vec{n}\times V_j(\vec{n}\psi))+V_j\vec{\psi}+\operatorname{grad}_T V_j v,\\ 2(-1)^j\operatorname{div}\vec{u}|_\Gamma &= ((-1)^j-K_j)\psi+V_j\operatorname{div}_T\vec{\psi}-k_j^2V_jv,\\ 2(-1)^j(-\vec{n}\times\operatorname{curl}\vec{u})|_\Gamma &= D_j\vec{v}-\vec{n}\times\operatorname{curl}V_j(\vec{n}\psi)+(-1)^j\vec{\psi}-\vec{n}\times\vec{K}_j(\vec{n}\times\vec{\psi}),\\ 2(-1)^j\vec{n}\,.\,\vec{u}|_\Gamma &= -\vec{n}\,.\operatorname{curl}V_j(\vec{n}\times\vec{v})+\vec{n}\,.\,V_j(\vec{n}\psi)+\vec{n}\,.\,V_j\vec{\psi}+((-1)^j+K_j')v.\end{aligned} \tag{4.7}$$

The right-hand side of (4.7) defines (up to a factor $2(-1)^j$) the Calderón projection operator $\tilde{C}_j$ (cf. Lemma 2.1 and equation (2.24)) for the problem (3.6), (3.18), (3.8). Thus it is a matrix of pseudodifferential operators whose principal symbol we shall now compute. As is known from the calculus of pseudodifferential operators, the principal symbol gives easy criteria for continuity of the operators in Sobolev spaces, for their ellipticity (Fredholm properties) and also strong ellipticity. By introducing a basis of orthonormal coordinates in the cotangent bundle $T^*(\Gamma)$ (see [**7**, p. 255]), for the pseudodifferential operators on the closed, smooth, bounded manifold Γ we obtain the same principal symbols as in the half-space case. Thus we may simply assume that Γ is a plane. Then the principal symbols of the pseudodifferential operators in (4.7) are easily obtained by use of Fourier transformation.

Let Ω_1 coincide with $\mathbb{R}_3^- := \{x\in\mathbb{R}^3 \mid x=(x_1,x_2,x_3), x_3<0\}$ and Ω with $\mathbb{R}_3^+ := \{x\in\mathbb{R}^3 \mid x_3>0\}$ and $\vec{n}=(0,0,1)$, yielding $\vec{n}\times\vec{a}=(-a_2,a_1)$ for any vector $\vec{a}=(a_1,a_2,a_3)\in\mathbb{R}^3$.

In the following lemma we list the principal symbols of those pseudodifferential operators on Γ which arise in our solution procedure. The orders of the operators are those given by Lemma 2.1, namely $-1, 0$, or $+1$, respectively. Thus, for example, the operator $\vec{K}_j$ has a vanishing principal symbol, because in our Agmon–Douglis–Nirenberg-elliptic system (4.7), $\vec{K}_j$ is considered as an operator of order 0, whereas it is in fact a pseudodifferential operator of order -1.

LEMMA 4.5. *For any $\xi\in\mathbb{R}^2$, $\xi\neq(0,0)$ with $|\xi|=\sqrt{\xi_1^2+\xi_2^2}$ for the principal symbols σ_m of order m there holds*:

$$\sigma_{-1}(V_j)(\xi)=-\frac{1}{|\xi|};\quad \sigma_0(K_j)(\xi)=0=\sigma_0(K_j')(\xi);\quad \sigma_0(\vec{n}\times\cdot)(\xi)=\begin{pmatrix}0&-1\\1&0\end{pmatrix};$$

$$\sigma_1(\operatorname{grad}_T)(\xi)=i\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix};\quad \sigma_1(\vec{n}\,.\operatorname{curl})(\xi)=i(-\xi_2,\xi_1);\quad \sigma_1(\operatorname{div}_T)(\xi)=i(\xi_1,\xi_2);$$

$$\sigma_1(D_j)(\xi)=-\frac{1}{|\xi|}\begin{pmatrix}\xi_2^2&-\xi_1\xi_2\\-\xi_1\xi_2&\xi_1^2\end{pmatrix};\quad \sigma_0(-\vec{n}\times\operatorname{curl}V_j\vec{n}\,\cdot)(\xi)=\frac{i}{|\xi|}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix};$$

$$\sigma_0(\vec{K}_j)(\xi)=\begin{pmatrix}0&0\\0&0\end{pmatrix};$$

$$\sigma_0(-\vec{n}\times \operatorname{curl} V_j\vec{n}\operatorname{div}_T)(\xi) = -\frac{1}{|\xi|}\begin{pmatrix}\xi_1^2 & \xi_1\xi_2\\ \xi_1\xi_2 & \xi_2^2\end{pmatrix}$$
$$= \sigma_0(-\operatorname{grad}_T \vec{n}\,.\operatorname{curl} V_j(\vec{n}\times\cdot))(\xi)$$
$$= -\sigma_0(-\operatorname{grad}_T \vec{n}\,.\, V_j\vec{n}\operatorname{div}_T)(\xi).$$

Proof. The general calculus of pseudodifferential operators ([**7**], [**27**]) shows that the pseudodifferential operators on a smooth manifold Γ have the same principal symbols as the corresponding operators of the half-space case. Furthermore, exchanging points of integration and restriction in the boundary integral operators causes perturbations of lower order only. In particular, we know from [**19**] that $W_j\vec{u} := \vec{n}V_j(\vec{n}\,.\,\vec{u}) - V_j(\vec{n}\,.\,\vec{n})\vec{u}$ defines a pseudodifferential operator of order -2. Similarly, $\sigma_{-1}(\vec{n}\times V_j(\vec{n}\,\cdot))(\xi) = \sigma_{-1}(V_j(\vec{n}\times(\vec{n}\,\cdot)))(\xi) = 0$.

From [**13**] and [**19**], $\sigma_{-1}(V_j)(\xi) = -1/|\xi|$ follows by taking the Fourier transform of the fundamental solution of the Laplacian $1/(4\pi\,|x-y|)$ which is the leading term in the Taylor series expansion of (4.2). From the identity $\Delta\vec{u} = \operatorname{grad}\operatorname{div}\vec{u} - \operatorname{curl}\operatorname{curl}\vec{u}$ we obtain, for any smooth tangential field $\vec{v}$,

$$D_j\vec{v} = k_j^2(\vec{n}\times V_j(\vec{n}\times\vec{v})) + \vec{n}\times\operatorname{grad} V_j\operatorname{div}_T(\vec{n}\times\vec{v}).$$

Thus

$$\sigma_1(D_j)(\xi) = \sigma_1(\vec{n}\times\operatorname{grad}_T)(\xi)\,.\,\sigma_{-1}(V_j)(\xi)\,.\,\sigma_1(\operatorname{div}_T(\vec{n}\times\cdot))(\xi)$$

yields the result for $\sigma_1(D_j)(\xi)$. The other symbols are computed similarly. □

Applying the general results of Section 2, we see that the Calderón projector from the system (4.7) has the form

$$\tilde{C}_j = \tfrac{1}{2}(1 + (-1)^j A_j), \tag{4.8}$$

where the operator A_j is given by

$$A_j\begin{pmatrix}\vec{v}\\ \psi\\ \vec{\psi}\\ v\end{pmatrix} = \begin{pmatrix} -\vec{K}_j\vec{v} + (V_j(\vec{n}\psi))_T + V_j\vec{\psi} + \operatorname{grad}_T V_j v\\ -K_j\psi + V_j\operatorname{div}_T\vec{\psi} - k_j^2 V_j v\\ D_j\vec{v} - \vec{n}\times\operatorname{curl} V_j(\vec{n}\psi) - \vec{n}\times\vec{K}_j(\vec{n}\times\vec{\psi})\\ -\vec{n}\,.\operatorname{curl} V_j(\vec{n}\times\vec{v}) + \vec{n}\,.\,V_j(\vec{n}\psi) + \vec{n}\,.\,V_j\vec{\psi} + K_j'v\end{pmatrix}. \tag{4.9}$$

The operator H from (2.25) is given by

$$H = \tilde{C}_1 - \tilde{C}_2 = -\tfrac{1}{2}(A_1 + A_2). \tag{4.10}$$

Hence with Lemma 4.5 its principal symbol is given by

$$\sigma(H)(\xi) = -\sigma(A_j)(\xi) = \frac{1}{|\xi|}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & i\xi_1\\ 0 & 0 & 0 & 0 & 1 & i\xi_2\\ 0 & 0 & 0 & i\xi_1 & i\xi_2 & 0\\ \xi_2^2 & -\xi_1\xi_2 & -i\xi_1 & 0 & 0 & 0\\ -\xi_1\xi_2 & \xi_1^2 & -i\xi_2 & 0 & 0 & 0\\ -i\xi_1 & -i\xi_2 & 1 & 0 & 0 & 0\end{pmatrix}. \tag{4.11}$$

Thus, $H\colon \mathcal{H}^s \to \mathcal{H}^s$ is continuous for any real s where

$$\mathcal{H}^s := TH^{s+\frac{1}{2}}(\Gamma) \times H^{s-\frac{1}{2}}(\Gamma) \times TH^{s-\frac{1}{2}}(\Gamma) \times H^{s+\frac{1}{2}}(\Gamma).$$

We remark that the off-diagonal blocks of $\sigma(H)(\xi)$ in (4.11), namely the matrices

$$E = \frac{1}{|\xi|}\begin{pmatrix} 1 & 0 & i\xi_1 \\ 0 & 1 & i\xi_2 \\ i\xi_1 & i\xi_2 & 0 \end{pmatrix}, \quad M = \frac{1}{|\xi|}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 & -i\xi_1 \\ -\xi_1\xi_2 & \xi_1^2 & -i\xi_2 \\ -i\xi_1 & -i\xi_2 & 1 \end{pmatrix}, \tag{4.12}$$

are also the principal symbols of the integral operators which arise via the direct method for the "electric" and "magnetic" boundary value problems, respectively **[25]**, **[24]**.

The natural duality on $\mathcal{H}^0$ is given by

$$\left(\begin{pmatrix} \vec{v} \\ \psi \\ \vec{\psi} \\ v \end{pmatrix}, \begin{pmatrix} \vec{w} \\ \chi \\ \vec{\chi} \\ w \end{pmatrix}\right)_{\mathcal{H}^0} := \int_\Gamma \{\bar{\vec{v}} \cdot \vec{\chi} + \bar{\psi} w + \bar{\vec{\psi}} \cdot \vec{w} + \bar{v}\chi\}\, do \tag{4.13}$$

for smooth elements in $\mathcal{H}^0$ (cf. (2.35)); this also corresponds to Green's formula (3.25).

Therefore, with respect to this sesquilinear form, the operator H in (4.10) is strongly elliptic, i.e. satisfies a Gårding inequality, if and only if the matrices E and M in (4.12) define positive definite quadratic forms on $\mathbb{C}^3$, i.e. their selfadjoint parts are positive definite matrices. That this is *not* the case can easily be seen as follows: from equations (4.11)–(4.13) we see that in the half-space case, i.e. on the symbol level, there holds for $\Psi := (\vec{v}, \psi, \vec{\psi}, v)^{\mathsf{T}}$,

$$(\Psi, H\Psi)_{\mathcal{H}^0} = \int \left\{\overline{(\vec{v}, \psi)} M \begin{pmatrix} \vec{v} \\ \psi \end{pmatrix} + \overline{(\vec{\psi}, v)} E \begin{pmatrix} \vec{\psi} \\ v \end{pmatrix}\right\} d\xi_1\, d\xi_2. \tag{4.14}$$

Furthermore, the selfadjoint parts of both M and E are singular matrices:

$$\tfrac{1}{2}(E + \overline{E^{\mathsf{T}}}) = \frac{1}{|\xi|}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tfrac{1}{2}(M + \overline{M^{\mathsf{T}}}) = \frac{1}{|\xi|}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 & 0 \\ -\xi_1\xi_2 & \xi_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the matrices E and M themselves are non-singular which corresponds to the *ellipticity* of the corresponding pseudodifferential operator. Likewise, the operator H in (4.10) is an elliptic pseudodifferential operator and hence a Fredholm operator in the space $\mathcal{H}^s$, $s \in \mathbb{R}$.

Thus, using standard arguments from the calculus of elliptic pseudodifferential operators, one can derive *a priori* estimates for H and thus obtain regularity results for the solution $\Psi \in \mathcal{H}^s$ of the system

$$H\Psi := -\tfrac{1}{2}(A_1 + A_2)\Psi = \tfrac{1}{2}(1 - A_2)\Psi_0 \tag{4.15}$$

for given $\Psi_0 \in \mathcal{H}^s$ with A_j in (4.9), $j = 1, 2$.

We note that equation (4.15) follows from (2.22)–(2.25) together with the Calderón projector (4.8). Therefore, application of Theorem 2.4 shows that the solution of (4.15) is the set of the Cauchy data of the refracted field in (3.6), (3.18) (see also [25], [4]).

According to the general results in [23], the least squares method for (4.15) – with regular finite elements on the interface manifold Γ – converges with quasi-optimal order. However, its convergence rate is considerably smaller compared with that of the Galerkin method. Therefore, we are more interested in a suitable Galerkin procedure. Unfortunately, H is not strongly elliptic with respect to the energy form $(\cdot,\cdot)_{\mathscr{H}^0}$ in (4.13), as we have shown above. But strong ellipticity is necessary and sufficient for the convergence of general Galerkin procedures [12], [28], [29], [30].

In order to obtain a strongly elliptic boundary integral operator for the transmission problem (3.6), (3.18), (3.8), we modify the Cauchy data as in (3.26) and insert them into the system (4.7). For the new Cauchy data

$$\Psi' := (\vec{v}, \psi', \vec{\psi}', v)^{\mathrm{T}} \in \mathscr{H}^0, \quad \psi' := \psi - \operatorname{div}_{\mathrm{T}} \vec{v}, \quad \vec{\psi}' := \vec{\psi} + \operatorname{grad}_{\mathrm{T}} v \tag{4.16}$$

with $\vec{v}$, ψ, $\vec{\psi}$, v as in Definition 4.1, the system (4.7) takes the form

$$\left.\begin{aligned}
2(-1)^j\vec{v} &= ((-1)^j - \vec{K}_j)\vec{v} + (V_j(\vec{n}\operatorname{div}_{\mathrm{T}}\vec{v}))_{\mathrm{T}} + (V_j(\vec{n}\psi'))_{\mathrm{T}}\\
&\quad + V_j\vec{\psi}' - V_j\operatorname{grad}_{\mathrm{T}} v + \operatorname{grad}_{\mathrm{T}} V_j v,\\
2(-1)^j\psi' &= L_j\vec{v} + ((-1)^j - K_j)\psi' - \operatorname{div}_{\mathrm{T}}(V_j\vec{n}\psi'))_{\mathrm{T}}\\
&\quad + V_j\operatorname{div}_{\mathrm{T}}\vec{\psi}' - \operatorname{div}_{\mathrm{T}} V_j\vec{\psi}' + M_j v,\\
2(-1)^j\vec{\psi}' &= N_j\vec{v} + \operatorname{grad}_{\mathrm{T}}\vec{n}\,.\,V_j(\vec{n}\psi') - \vec{n}\times\operatorname{curl} V_j(\vec{n}\psi')\\
&\quad + (-1)^j\vec{\psi}' - \vec{n}\times\vec{K}_j(\vec{n}\times\vec{\psi}') + \operatorname{grad}_{\mathrm{T}}\vec{n}\,.\,V_j\vec{\psi}' + R_j v,\\
2(-1)^j v &= -\vec{n}\,.\operatorname{curl} V_j(\vec{n}\times\vec{v}) + \vec{n}\,.\,V_j(\vec{n}\operatorname{div}_{\mathrm{T}}\vec{v})\\
&\quad + \vec{n}\,.\,V_j(\vec{n}\psi') + \vec{n}\,.\,V_j\vec{\psi}' - \vec{n}\,.\,V_j\operatorname{grad}_{\mathrm{T}} v + ((-1)^j + K_j')v,
\end{aligned}\right\}\tag{4.17}$$

with

$$\left.\begin{aligned}
L_j\vec{v} &:= \operatorname{div}_{\mathrm{T}} K_j\vec{v} - K_j\operatorname{div}_{\mathrm{T}}\vec{v} - \operatorname{div}(V_j(\vec{n}\operatorname{div}_{\mathrm{T}}\vec{v}))_{\mathrm{T}},\\
M_j v &:= \operatorname{div}_{\mathrm{T}} V_j\operatorname{grad}_{\mathrm{T}} v - V_j\operatorname{div}_{\mathrm{T}}\operatorname{grad}_{\mathrm{T}} v - k_j^2 V_j v - \operatorname{div}_{\mathrm{T}}\operatorname{grad}_{\mathrm{T}} V_j v,\\
N_j\vec{v} &:= D_j\vec{v} - \vec{n}\times\operatorname{curl} V_j(\vec{n}\operatorname{div}_{\mathrm{T}}\vec{v})\\
&\quad + \operatorname{grad}_{\mathrm{T}}(-\vec{n}\,.\operatorname{curl} V_j(\vec{n}\times\vec{v})) + \operatorname{grad}_{\mathrm{T}}\vec{n}\,.\,V_j(\vec{n}\operatorname{div}_{\mathrm{T}}\vec{v}),\\
R_j v &:= \vec{n}\times K_j(\vec{n}\times\operatorname{grad}_{\mathrm{T}} v) - \operatorname{grad}_{\mathrm{T}}\vec{n}\,.\,V_j\operatorname{grad}_{\mathrm{T}} v + \operatorname{grad}_{\mathrm{T}} K_j' v.
\end{aligned}\right\}\tag{4.18}$$

Therefore the operator H' from (2.25) has the form

$$H' = -\tfrac{1}{2}(A_1' + A_2'), \tag{4.19}$$

where

$$A_j'\begin{pmatrix}\vec{v}\\ \psi'\\ \vec{\psi}'\\ v\end{pmatrix}$$

$$=\begin{pmatrix} -\vec{K}_j\vec{v}+(V_j(\vec{n}\operatorname{div}_T\vec{v}))_T+(V_j(\vec{n}\psi'))_T+V_j\vec{\psi}'-V_j\operatorname{grad}_T v+\operatorname{grad}_T V_j v\\ L_j\vec{v}-K_j\psi'-\operatorname{div}_T(V_j\vec{n}\psi'))_T+V_j\operatorname{div}_T\vec{\psi}'-\operatorname{div}_T V_j\vec{\psi}'+M_j v\\ N_j\vec{v}+\operatorname{grad}_T\vec{n}\,.\,V_j(\vec{n}\psi')-\vec{n}\times\operatorname{curl}V_j(\vec{n}\psi')\\ \qquad -\vec{n}\times\vec{K}_j(\vec{n}\times\vec{\psi}')+\operatorname{grad}_T\vec{n}\,.\,V_j\vec{\psi}'+R_j v\\ -\vec{n}\,.\operatorname{curl}V_j(\vec{n}\times\vec{v})+\vec{n}\,.\,V_j(\vec{n}\operatorname{div}_T\vec{v})+\vec{n}\,.\,V_j(\vec{n}\psi')\\ \qquad +\vec{n}\,.\,V_j\vec{\psi}'-\vec{n}\,.\,V_j\operatorname{grad}_T v+K_j'v \end{pmatrix}. \tag{4.20}$$

Hence with Lemma 4.5 and equation (4.11) the principal symbol of H' is given by

$$\sigma(H')(\xi)=\frac{1}{|\xi|}\begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&|\xi|^2\\ |\xi|^2&0&0&0&0&0\\ 0&|\xi|^2&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}. \tag{4.21}$$

The principal symbol shows that H' is continuous from $\mathscr{H}^s$ into itself for any real s. Furthermore, its off-diagonal blocks

$$E'=\begin{pmatrix}|\xi|^{-1}&0&0\\ 0&|\xi|^{-1}&0\\ 0&0&|\xi|\end{pmatrix},\quad M'=\begin{pmatrix}|\xi|&0&0\\ 0&|\xi|&0\\ 0&0&|\xi|^{-1}\end{pmatrix}, \tag{4.22}$$

obviously define positive quadratic forms on $\mathbb{C}^3$. We note that E' and M' are used in **[25]**, **[24]** to derive the "edge behaviour" of the "electric" and the "magnetic" fields, respectively, since the components of the fields are decoupled in first order, i.e. they are only coupled via compact perturbations in (4.20).

From standard results on pseudodifferential operators **[27]**, **[30]** we deduce that the form (4.21) of the principal symbol $\sigma(H')$ implies the coerciveness of H' in the sense of a Gårding inequality in $\mathscr{H}^0$.

LEMMA 4.6. *There exists a real $\gamma>0$ such that (with the duality* (4.13))

$$\operatorname{Re}(\Psi, H'\Psi)_{\mathscr{H}^0}\geqq\gamma\,\|\Psi\|^2_{\mathscr{H}^0}-|k(\Psi,\Psi)| \tag{4.23}$$

for all $\Psi\in\mathscr{H}^0$ with a compact bilinear form $k(\cdot,\cdot)$ on $\mathscr{H}^0\times\mathscr{H}^0$.

Proof. The arguments following (4.13) show that in the half-space case (4.14) holds with E' and M' instead of E and M yielding (4.23) due to (4.22). The compact bilinear form $k(\cdot, \cdot)$ arises in (4.23) since H' acts on functions on a bounded manifold Γ which causes a compact perturbation to the half-space situation. □

Now we concentrate on the connection between our strongly elliptic pseudodifferential operator H' and the original interface problem (3.6), (3.18), (3.8). Following (2.22)–(2.25), for (3.6), (3.18), (3.8) we obtain the system of integral equations

$$H'\Psi' = -(1 - \tilde{C}_2')\Psi_0' \quad \text{with} \quad H' := \tilde{C}_1' - \tilde{C}_2', \tag{4.24}$$

where $\Psi_0' := (\vec{v}_0, \psi_0', \vec{\psi}', v_0)^{\mathrm{T}}$ are the modified Cauchy data (4.16) of the incident field $\vec{u}_0$ and $\tilde{C}_j' = \frac{1}{2}(1 + (-1)^j A_j')$ is the Calderón projector (corresponding to the wave number k_j) of A_j' in (4.20). Application of Theorem 2.4 to (4.24) yields the following equivalence between the transmission problem (3.6), (3.18), (3.8) and the boundary integral equations (4.24). (The proof is identical to the proof of Theorem 2.4 and is therefore omitted.)

THEOREM 4.7. *Let $\vec{u}_0 \in \mathcal{H}^s$ be given.*

(i) *If $\vec{u}_j \in L_j$ $(j = 1, 2)$ as defined in* (4.1) *solve the transmission problem* (3.6), (3.18), (3.8), *then*

$$((\vec{u}_1)_{\mathrm{T}}, \operatorname{div} \vec{u}_1 - \operatorname{div}_{\mathrm{T}} (\vec{u}_1)_{\mathrm{T}}, -\vec{n} \times \operatorname{curl} \vec{u}_1 + \operatorname{grad}_{\mathrm{T}} (\vec{n} \cdot \vec{u}_1), \vec{n} \cdot \vec{u}_1)^{\mathrm{T}} \in \mathcal{H}^s$$

solves the equation (4.24).

(ii) *If $\Psi' := (\vec{v}, \psi', \vec{\psi}', v)^{\mathrm{T}} \in \mathcal{H}^s$ solves* (4.24) *with $\Psi_0' := (\vec{v}_0, \psi_0', \vec{\psi}_0', v_0)^{\mathrm{T}}$, i.e.*

$$-\tfrac{1}{2}(A_1' + A_2')\Psi' = \tfrac{1}{2}(A_2' - 1)\Psi_0',$$

then with

$$\Psi_1' := \tilde{C}_1'\Psi', \quad \Psi_2' := \tilde{C}_2'(\Psi' + \Psi_0') \tag{4.25}$$

and

$$\vec{u}_j := K_j \mathcal{P}_j R_j^{-1} \Psi_j' \quad \text{in } \Omega_j \quad (\text{see } (2.18)), \tag{4.26}$$

$\vec{u}_j \in L_j$ solve the transmission problem (3.6), (3.18), (3.8).

Remark. In (4.26), $\vec{u}_j$ is given by the Stratton–Chu formula (4.4) applied to $(\vec{v}_j, \psi_j, \vec{\psi}_j, v_j)$ which are connected with Ψ_j' via (4.16), (3.27), (3.28). By Gårding's inequality (4.23), H' is a Fredholm operator of index zero from $\mathcal{H}^0$ into itself and thus also from $\mathcal{H}^s$ into itself for any s. Therefore we obtain existence of a solution of (4.24) as soon as we know its uniqueness, and Theorem 4.7 then implies the existence of a solution of the transmission problem. For the question of unique solvability of the transmission problem (3.6), (3.18), (3.8) and therefore of our boundary integral equation (4.24) we refer to **[2, 4, 15, 20, 25]**. In the case of the "physical" transmission conditions (3.7), we discuss this question in more detail in Section 5. From the discussion in **[4]** and **[25]** we have the following result:

PROPOSITION 4.8. *Assume that the homogeneous transmission problem* (3.6), (3.18), (3.8) *and an associated homogeneous adjoint problem with interchanged wave numbers have only the trivial solution in L_j (defined in* (4.1)*). Then for given $\Psi_0 \in \mathcal{H}^s$ there exists exactly one solution $\Psi' \in \mathcal{H}^s$ of the integral equation* (4.24) *yielding exactly one solution of* (3.6), (3.18), (3.8) *via* (4.26).

5. Integral equations for the electromagnetic transmission problem

In this section we study the integral equations corresponding to the "physical" transmission conditions (3.7). Instead of repeating all the arguments of the preceding section, we only point out the necessary modifications.

We define, according to (3.7), the "physical" Cauchy data on Γ:

$$\left.\begin{aligned} &\tilde{v}_j := \varepsilon_j v_j = \varepsilon_j \vec{n} \,.\, \vec{u}_j; \quad \vec{\tilde{v}}_j = \vec{u}_{j\mathrm{T}}; \\ &\tilde{\psi}_j := \lambda_j \psi_j = \lambda_j \operatorname{div} \vec{u}_j; \quad \vec{\tilde{\psi}}_j := \frac{1}{\mu_j} \vec{\psi}_j = -\frac{1}{\mu_j} \times \operatorname{curl} \vec{u}_j. \end{aligned}\right\} \tag{5.1}$$

We want to use the results of the previous section. Therefore we write

$$\Psi := (\vec{v}, \psi, \vec{\psi}, v)^{\mathrm{T}}; \quad \tilde{\Psi}_j := (\vec{\tilde{v}}_j, \tilde{\psi}_j, \vec{\tilde{\psi}}_j, \tilde{v}_j)^{\mathrm{T}}.$$

Thus, with the obvious block notation, we have

$$\tilde{\Psi}_j = B_j \Psi; \quad B_j = \operatorname{diag}\left(1, \lambda, \frac{1}{\mu_j}, \varepsilon_j\right). \tag{5.2}$$

Now we can insert these Cauchy data into the representation formula as before and obtain, instead of (4.8), the Calderón projectors

$$\tilde{C}_j = \tfrac{1}{2}(1 + (-1)^j \tilde{A}_j); \quad \tilde{A}_j = B_j A_j B_j^{-1}. \tag{5.3}$$

The boundary integral operators A_j are given explicitly in (4.9). The procedure described in the previous section then leads to a boundary integral equation corresponding to (2.25). The matrix of integral operators is given by

$$\tilde{H} = \tilde{C}_1 - \tilde{C}_2 = -\tfrac{1}{2}(\tilde{A}_1 + \tilde{A}_2) = -\tfrac{1}{2}(B_1 A_1 B_1^{-1} + B_2 A_2 B_2^{-1}). \tag{5.4}$$

From Lemma 4.5 and (4.11) we find the principal symbol (in block form)

$$\sigma(\tilde{H})(\xi) = \begin{pmatrix} 0 & \tilde{E} \\ \tilde{M} & 0 \end{pmatrix} \tag{5.5}$$

with

$$\tilde{E} = \frac{1}{|\xi|} \begin{pmatrix} \mu_e & 0 & i\xi_1 \varepsilon_e \\ 0 & \mu_e & i\xi_2 \varepsilon_e \\ i\xi_1 \lambda_e & i\xi_2 \lambda_e & 0 \end{pmatrix}, \quad \tilde{M} = \frac{1}{|\xi|} \begin{pmatrix} \xi_2^2 \mu_m & -\xi_1 \xi_2 \mu_m & -i\xi_1 \lambda_m \\ -\xi_1 \xi_2 \mu_m & \xi_1^2 \mu_m & -i\xi_2 \lambda_m \\ -i\xi_1 \varepsilon_m & -i\xi_2 \varepsilon_m & \nu_m \end{pmatrix}. \tag{5.6}$$

Here we used the abbreviations

$$\left.\begin{aligned} &\mu_e = \tfrac{1}{2}(\mu_1 + \mu_2); \quad \varepsilon_e = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right); \quad \lambda_e = \tfrac{1}{2}(\lambda_1 \mu_1 + \lambda_2 \mu_2); \quad \nu_e = \frac{1}{2}\left(\frac{\lambda_1}{\varepsilon_1} + \frac{\lambda_2}{\varepsilon_2}\right); \\ &\mu_m = \frac{1}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right); \quad \varepsilon_m = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2); \quad \lambda_m = \frac{1}{2}\left(\frac{1}{\lambda_1 \mu_1} + \frac{1}{\lambda_2 \mu_2}\right); \quad \nu_m = \frac{1}{2}\left(\frac{\varepsilon_1}{\lambda_1} + \frac{\varepsilon_2}{\lambda_2}\right). \end{aligned}\right\} \tag{5.7}$$

Again, as in the previous section, the matrices $\tilde{E}$ and $\tilde{M}$ are, in general, not positive, hence the operator $\tilde{H}$ will not be strongly elliptic. Therefore we modify the Cauchy data (5.1) analogously to (3.26), (4.16):

$$\tilde{\Psi}'_j := (\vec{\tilde{v}}_j, \tilde{\psi}'_j, \vec{\tilde{\psi}}'_j, \tilde{v}'_j)^{\mathsf{T}}$$

with

$$\left.\begin{aligned}
&\tilde{\psi}'_j := \eta\tilde{\psi}_j - \operatorname{div}_{\mathsf{T}} \vec{\tilde{v}}_j = \eta\lambda_j\psi'_j + (\eta\lambda_j - 1)\operatorname{div}_{\mathsf{T}} \vec{v}_j = \eta\lambda_j\psi_j - \operatorname{div}_{\mathsf{T}} \vec{v}_j;\\
&\vec{\tilde{\psi}}'_j := \vec{\tilde{\psi}}_j + \vartheta \operatorname{grad}_{\mathsf{T}} \tilde{v}_j = \frac{1}{\mu_j}\vec{\psi}'_j + \left(\vartheta\varepsilon_j - \frac{1}{\mu_j}\right)\operatorname{grad}_{\mathsf{T}} v_j = \frac{1}{\mu_j}\vec{\psi}_j + \vartheta\varepsilon_j \operatorname{grad}_{\mathsf{T}} v_j;\\
&\tilde{v}'_j := \vartheta\tilde{v}_j = \vartheta\varepsilon_j v_j.
\end{aligned}\right\} \quad (5.8)$$

Here we introduce two new complex parameters η and ϑ which will be fixed later on (see (5.11) and (5.13)). It turns out that they can always be chosen in such a way that the resulting boundary integral operator is strongly elliptic.

It is obvious how to insert these constants into the system of integral equations (4.17). Therefore we need not repeat this explicit representation here. In short notation, the modified system of integral operators is

$$\tilde{H}' = \tilde{C}'_1 - \tilde{C}'_2 = -\tfrac{1}{2}(\tilde{A}'_1 + \tilde{A}'_2) = -\tfrac{1}{2}(B'_1A'_1B_1'^{-1} + B'_2A'_2B_2'^{-1}), \quad (5.9)$$

with A'_j as defined in (4.20) and, according to (5.8),

$$B'_j = \begin{pmatrix} 1 & 0 & 0 & 0\\ (\eta\lambda_j - 1)\operatorname{div}_{\mathsf{T}} & \eta\lambda_j & 0 & 0\\ 0 & 0 & \mu_j^{-1} & (\vartheta\varepsilon_j - \mu_j^{-1})\operatorname{grad}_{\mathsf{T}}\\ 0 & 0 & 0 & \vartheta\varepsilon_j \end{pmatrix}$$

with the obvious block notation.

For the computation of the principal symbols, we can take advantage of (4.21), (4.22). The result is

$$\sigma(\tilde{H}')(\xi) = \begin{pmatrix} 0 & \tilde{E}'\\ \tilde{M}' & 0 \end{pmatrix},$$

with $\tilde{E}' = \frac{1}{2}(\tilde{E}'_1 + \tilde{E}'_2)$; $\tilde{M}' = \frac{1}{2}(\tilde{M}'_1 + \tilde{M}'_2)$ and

$$\tilde{E}'_j = \frac{1}{|\xi|}\begin{pmatrix} \mu_j & 0 & ia\xi_1\\ 0 & \mu_j & ia\xi_2\\ ib\xi_1 & ib\xi_2 & c\,|\xi|^2 \end{pmatrix}; \quad \tilde{M}'_j = (\tilde{E}'_j)^{-1},$$

where

$$a = (\vartheta\varepsilon_j)^{-1} - \mu_j; \quad b = \mu_j(\eta\lambda_j - 1); \quad c = a + b + \mu_j.$$

For simplicity, we now make the choices,

$$\lambda_j = (\mu_j\bar{\varepsilon}_j)^{-1} \quad \text{(compare case (ii) of Section 3)}, \quad (5.10)$$

and

$$\vartheta = (\bar{\eta})^{-1}. \quad (5.11)$$

This gives $b = \bar{a}$, $c = \bar{c}$; hence, with $d_j = 2\operatorname{Re}(\bar{\eta}\varepsilon_j^{-1}) - \mu_j$, $e_j = |\mu_j\bar{\varepsilon}_j\eta^{-1} - 1|^2$, one obtains

$$\tfrac{1}{2}(\tilde{E}_j' + \overline{(\tilde{E}_j')^{\mathrm{T}}}) = \frac{1}{|\xi|}\begin{pmatrix} \mu_j & 0 & 0 \\ 0 & \mu_j & 0 \\ 0 & 0 & d_j\,|\xi|^2 \end{pmatrix};$$

$$\tfrac{1}{2}(\tilde{M}_j' + \overline{(\tilde{M}_j')^{\mathrm{T}}}) = \frac{1}{\mu_j\,|\xi|}\begin{pmatrix} |\xi|^2 - e_j\xi_1^2 & -e_j\xi_1\xi_2 & 0 \\ -e_j\xi_1\xi_2 & |\xi|^2 - e_j\xi_2^2 & 0 \\ 0 & 0 & |\mu_j\varepsilon_j/\eta|^2 \end{pmatrix}.$$

Because we assumed $\mu_j > 0$, the positivity of both matrices depends only on the positivity of d_j.

LEMMA 5.1. *The operator $\tilde{H}'$ is strongly elliptic, i.e. there exists a $\gamma > 0$ such that*

$$\operatorname{Re}(\Psi, \tilde{H}'\Psi)_{\mathscr{H}^0} \geqq \gamma\,\|\Psi\|^2_{\mathscr{H}^0} - |k(\Psi, \Psi)| \tag{5.12}$$

for all $\Psi \in \mathscr{H}^0$ with a compact bilinear form $k(\cdot, \cdot)$ on $\mathscr{H}^0 \times \mathscr{H}^0$, if and only if

$$\left.\begin{aligned} &2\operatorname{Re}\bar{\eta}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) > \mu_1 + \mu_2 \text{ and} \\ &2\operatorname{Re}\eta(\varepsilon_1 + \varepsilon_2) > \mu_1\,|\varepsilon_1|^2 + \mu_2\,|\varepsilon_2|^2. \end{aligned}\right\} \tag{5.13}$$

Remark. Condition (5.13) can always be satisfied by a suitable choice of η, under assumption (3.4) and also, for the eddy current problem (see case (iv) in Section 3), even by a large enough real η.

Let us write the system of integral equations as

$$\tilde{H}'\tilde{\Psi}' = -(1 - \tilde{C}_2')\Psi_0'. \tag{5.14}$$

We summarise the results of this section in the following theorem.

THEOREM 5.2. *Let the assumptions* (3.4), (5.10), (5.11), *and* (5.13) *for the coefficients be satisfied. Then for given $\Psi_0 := (\vec{v}_0, \psi_0, \vec{\psi}_0, v_0)^{\mathrm{T}} \in \mathscr{H}^0$, the transmission problem* (3.6)–(3.8) *has a unique solution $\vec{u}$ with $\vec{u}_j \in L_j$, $j = 1, 2$. This solution corresponds via* (5.1), (5.8) *to the unique solution $\tilde{\Psi}' \in \mathscr{H}^0$ of the system* (5.14) *of boundary integral equations. The boundary integral equations are a strongly elliptic system of pseudodifferential equations on* Γ.

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