The volume integral equation in time-harmonic dielectric scattering

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The Volume Integral Equation in dielectric scattering

$\Omega \subset \mathbb{R}^3$: bounded domain

$\eta \in C^1(\overline{\Omega})$: “dielectric contrast”

$g_k(x) = \dfrac{e^{ik|x|}}{4\pi|x|}$: fundamental solution of Helmholtz equation

The VIE in Ω

$$u(x) - \nabla_x \int_\Omega \eta(y) \nabla_y g_k(x-y) \cdot u(y)\,dy + k^2 \int_\Omega \eta(y) g_k(x-y) u(y)\,dy = f(x)$$

$$(1 - A_\eta)u = f$$

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Literature

1. Numerical methods:
J. P. Kottmann, O. J. F. Martin (2000)
C. C. Lu (2003)
M. M. Botha (2006)
M. I. Sancer, K. Sertel, J. L. Volakis, P. V. Alstine (2006)

2. Discussion of the spectrum:
J. Rahola (2000)
N. V. Budko, A. B. Samokhin (2006)

3. Theory, in $H(\mathbf{curl}, \Omega)$:
A. Kirsch (2007)

4. Collaboration in Rennes:
PhD thesis El Hadji Koné (2005–)
E. Darrigrand.

The VIO

$$A_\eta u(x) = \nabla_x \int_\Omega \nabla_y g_k(x-y)\cdot(\eta u)(y)dy - k^2 \int_\Omega g_k(x-y)(\eta u)(y)dy$$

The Problem: Spektrum of A_η ?

Known: For "physically reasonable" material coefficients and domains, the integral equation always has a unique solution.

Simple observations:

A_η maps boundedly :	$L^2(\Omega)$	to	$L^2(\Omega)$
(not compactly !)	$H(\mathbf{curl},\Omega)$	to	$H(\mathbf{curl},\Omega)$
	$H(\mathrm{div},\Omega)$	to	$H(\mathrm{div},\Omega)$

Lemma 1

$$\forall u \in L^2(\Omega): \qquad \mathrm{div}\, A_\eta u = \mathrm{div}(\eta u) \qquad \text{in } \Omega$$

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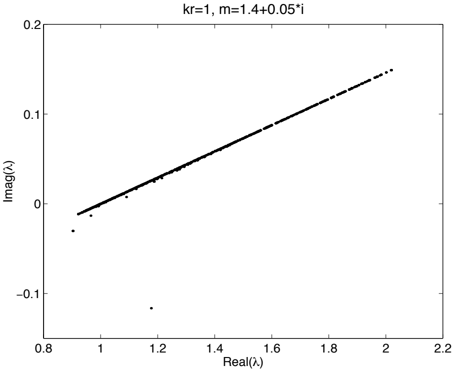

FIG. 3.1. *Eigenvalues of the coefficient matrix for a spherical scatterer of radius* $kr = 1$ *and refractive index* $m = 1.4 + 0.05i$*. The sphere is discretized with* 136 *computational cells (upper) and* 480 *computational cells (lower).*

$$\eta = 1 - m^2 = -0.9575 - 0.14i : \quad \text{line } \sim 1 - \eta.[0,1]$$

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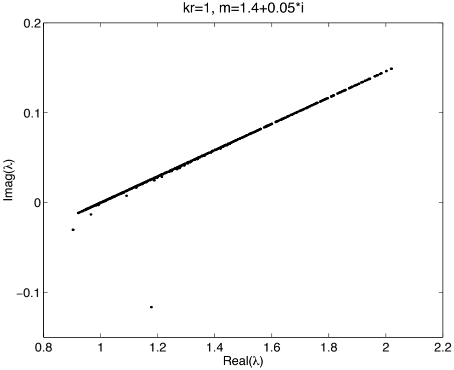

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Conjecture (for $\eta = 1$)

$$\sigma_{\text{ess}}(A_1) = [0, 1]$$

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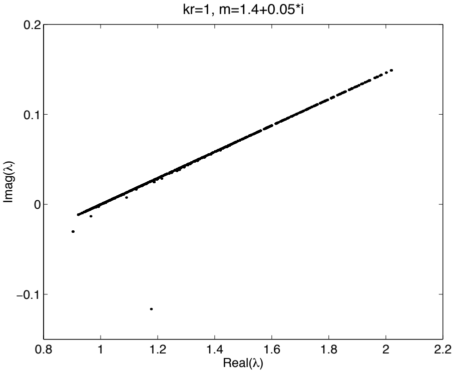

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Theorem (for $\eta = 1$, Ω regular)

$$\sigma_{\text{ess}}(A_1) = \left\{0, \tfrac{1}{2}, 1\right\}$$

Commutators etc:

$$\begin{aligned} A_\eta u(x) &= \nabla_x \int_\Omega \nabla_y g_k(x-y) \cdot u(y) \eta(y) dy - k^2 \int_\Omega g_k(x-y) u(y) \eta(y) dy \\ &= \eta(x) \nabla_x \int_\Omega \nabla_y g_0(x-y) \cdot u(y) dy + K u(x) \\ &= (\eta A + K) u(x), \qquad K : L^2(\Omega) \to L^2(\Omega) \text{ compact} \end{aligned}$$

$$A_\eta \approx \eta A \quad \text{with} \quad A = A_{\{\eta \equiv 1; k=0\}}$$

From now on: Study spectral theory of A in $L^2(\Omega)$.

$$Au(x) = \nabla_x \int_\Omega \nabla_y \frac{1}{4\pi|x-y|} \cdot u(y) dy$$

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Heuristics: “Aether waves”

$$Au(x) = \nabla_x \int_\Omega \nabla_y \frac{1}{4\pi|x-y|} \cdot u(y)dy$$

$$\Rightarrow (\lambda - A)u(x) = (-\lambda\Delta + \nabla \operatorname{div}) \int_\Omega \frac{1}{4\pi|x-y|} u(y)dy$$

Similar to COSSERAT eigenvalue problem in elasticity, with boundary conditions of the Newton potential (exterior Calderón projector).

The differential operator $-\lambda\Delta + \nabla\operatorname{div} = \lambda\,\mathbf{curl}\,\mathbf{curl} + (1-\lambda)\nabla\operatorname{div}$:

$\lambda = 0$:	$\nabla\operatorname{div}$:	not elliptic
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all other λ :		elliptic, but...
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Orthogonal decompositions:

$$L^2(\Omega)^3 = \nabla H_0^1(\Omega) \oplus V\,; \qquad V = H(\operatorname{div} 0, \Omega)$$

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S : single layer potential

Isomorphisms: $W \ni u \mapsto \gamma_n u = n \cdot u\big|_{\partial\Omega} \in H_*^{-1/2}(\partial\Omega)$

$$A\big|_W \leftrightarrow \partial_n S = \left(\tfrac{1}{2} - K'\right)\Big|_{H_*^{-1/2}(\partial\Omega)}$$

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Theorem 1

- If Ω is smooth, then

$$\sigma_{\text{ess}}(A_\eta) = \{0\} \cup \{\eta(x) \mid x \in \overline{\Omega}\} \cup \{\frac{\eta(x)}{2} \mid x \in \partial\Omega\}$$

- If Ω is Lipschitz, then there exists $J \subset\subset (0,1)$ such that

$$\sigma_{\text{ess}}(A_\eta) \subset \{0\} \cup \{\eta(x) \mid x \in \overline{\Omega}\} \cup \{\eta(x)r \mid x \in \partial\Omega, r \in J\}$$

OK, but...

Suitable for discretizations?

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The Essential Spectrum

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Proof: Fourier transformation of extension by zero $\tilde{u}$.

$$\begin{aligned}
(u, Au) &= \int\int_{\mathbb{R}^3} \tilde{u}(x) \nabla_x \nabla_y g_0(x-y) \cdot \overline{\tilde{u}(y)} dy dx \\
&= \int_{\mathbb{R}^3} \mathscr{F}\tilde{u}(\xi)^\top \frac{\xi\xi^\top}{|\xi|^2} \overline{\mathscr{F}\tilde{u}(\xi)} d\xi \\
&= \int_{\mathbb{R}^3} \left| \frac{\xi}{|\xi|} \cdot \mathscr{F}\tilde{u}(\xi) \right|^2 d\xi \\
&\le \int_{\mathbb{R}^3} |\mathscr{F}\tilde{u}(\xi)|^2 d\xi = \|u\|^2
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Theorem 2

Let $\eta(x) = 1 - \varepsilon_r(x) \in C^1(\overline{\Omega})$, $\mathrm{Re}\,\varepsilon_r(x) \ge \varepsilon_1 > 0$ $(\forall x \in \overline{\Omega})$.

Then there exist $c > 0$, $K : L^2(\Omega) \to L^2(\Omega)$ compact, such that

$$\mathrm{Re}\big(u, (1 - A_\eta)u\big) \ge c\,\|u\|^2 - (u, Ku)$$

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Theorem 2

Let $\eta(x) = 1 - \varepsilon_r(x) \in C^1(\overline{\Omega})$, $\mathrm{Re}\,\varepsilon_r(x) \ge \varepsilon_1 > 0$ ($\forall x \in \overline{\Omega}$).

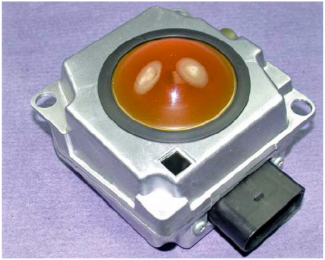
Then there exist $c > 0$, $K : L^2(\Omega) \to L^2(\Omega)$ compact, such that

$$\mathrm{Re}\big(u, (1 - A_\eta)u\big) \ge c\,\|u\|^2 - (u, Ku)$$

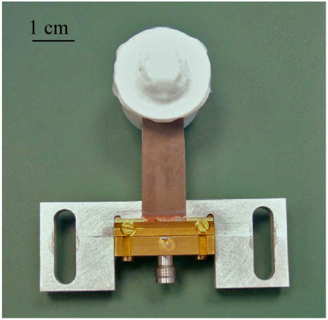
Proof: Set $\varepsilon^-(x) = \min\{1, \mathrm{Re}\,\varepsilon_r(x)\}$, $u_1 = \sqrt{\mathrm{Re}\,\varepsilon_r - \varepsilon^-}\,u$, $u_2 = \sqrt{1 - \varepsilon^-}\,u$.

$$\begin{aligned}
\mathrm{Re}\big(u, (1 - A_\eta)u\big) &\approx \big(u, (1 - (1 - \mathrm{Re}\,\varepsilon_r)A)u\big) \\
&= \big(u, \varepsilon^- u\big) + \big(u, (\mathrm{Re}\,\varepsilon_r - \varepsilon^-)A)u\big) + \big(u, (1 - \varepsilon^-)(1 - A)u\big) \\
&\approx \big(u, \varepsilon^- u\big) + \big(u_1, Au_1\big) + \big(u_2, (1 - A)u_2\big) \\
&\ge \big(u, \varepsilon^- u\big) \\
&\ge c\,\|u\|^2, \quad c = \min\{1, \varepsilon_1\}
\end{aligned}$$

Motivation: Some dielectric scatterers



ILA pour communication indoor à 62GHz.
Source : IST



. 4 : ILA compacte pour communication par
satellite à 49GHz [7]. Source : IETR

Motivation: Shape optimisation of dielectric lens

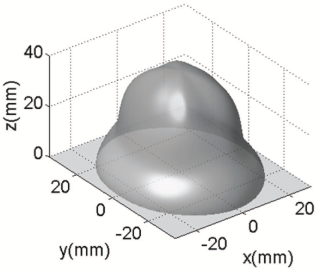


FIGURE 6. Optimized shape of the lens for a flat-top illumination.

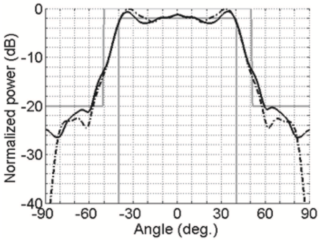
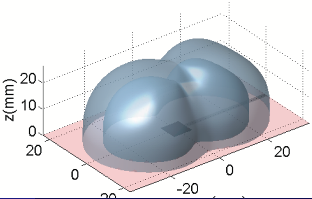
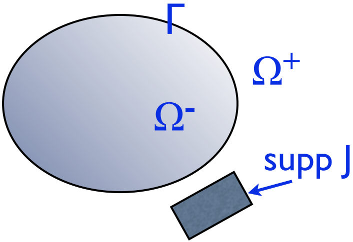


FIGURE 7. Computed radiation patterns in both principal planes at 28GHz. Solid grey line: power template. Solid and dotted black lines: co-polarization components in E- and H-planes.





Maxwell equations in $\mathbb{R}^3$

$$\begin{aligned} \mathbf{curl}\, E - i\,k\,H &= 0 \\ \mathbf{curl}\, H + i\,k\,\varepsilon_r\, E &= J \end{aligned}$$

Relative permittivity $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0}$. In Ω^+: $\varepsilon_r = 1$.

Def: $\eta := 1 - \varepsilon_r \Longrightarrow \operatorname{supp} \eta \subset \overline{\Omega^-}$

$k = \omega\sqrt{\varepsilon_0 \mu_0}$; $\mu \equiv \mu_0$ in $\mathbb{R}^3$.

- $J \in H(div)$
- $E, H \in L^2_{\text{loc}}$
- radiation condition

Jumps on Γ: $[n \times E] = 0 = [n \times H]$; $[n \cdot H] = 0$; $[n \cdot \varepsilon E] = 0$

The Lippmann-Schwinger equation [according to Colton-Kress '98]

Helmholtz equation in $\mathbb{R}^3$ with variable $k(x)$: $k(x) \equiv k = \text{const in } \Omega^+$

$$\begin{aligned}
(\Delta + k(x)^2)u &= f \\
(\Delta + k^2)u &= f - (k(x)^2 - k^2)u =: f - \kappa u \\
-u &= g_k * (f - \kappa u) \\
u - g_k * (\kappa u) &= -g_k * f \\
g_k(x) &= \frac{e^{ik|x|}}{4\pi|x|}
\end{aligned}$$

2nd kind weakly singular integral equation in Ω^-

$$u(x) - \int_{\Omega^-} g_k(x-y)\kappa(y)u(y)dy = -\int_{\Omega^-} g_k(x-y)f(y)dy$$

Maxwell → Helmholtz ("dyadic Green function")

$$-(\Delta + k^2) = \frac{1}{k^2}\left(\nabla \operatorname{div} + k^2\right)\left(\mathbf{curl}\,\mathbf{curl} - k^2\right)$$

$$\begin{aligned}
\mathbf{curl}\,\mathbf{curl}\,E - k^2 \varepsilon_r E &= ikJ \\
\mathbf{curl}\,\mathbf{curl}\,E - k^2 E &= ikJ - k^2 \eta E \\
-(\Delta + k^2)E &= -\tfrac{1}{ik}\nabla \operatorname{div} J + ikJ - (\nabla \operatorname{div} + k^2)(\eta E) \\
E &= F - g_k * (\nabla \operatorname{div} + k^2)(\eta E)
\end{aligned}$$

2nd kind strongly singular integral equation in $\Omega = \Omega$

$$E - \nabla \operatorname{div} g_k * (\eta E) + k^2 g_k * (\eta E) = F$$

(VIE)

$$E(x) - \nabla_x \int_{\Omega^-} \nabla_y g_k(x-y) \cdot E(y)\eta(y)\,dy + k^2 \int_{\Omega^-} g_k(x-y)E(y)\eta(y)\,dy = F(x)$$

Maxwell $\longrightarrow$ Helmholtz ("dyadic Green function")

$$-(\Delta + k^2) = \frac{1}{k^2}(\nabla \operatorname{div} + k^2)(\mathbf{curl}\,\mathbf{curl} - k^2)$$

$$\begin{aligned}
\mathbf{curl}\,\mathbf{curl}\, E - k^2 \varepsilon_r E &= ikJ \\
\mathbf{curl}\,\mathbf{curl}\, E - k^2 E &= ikJ - k^2 \eta E \\
-(\Delta + k^2) E &= -\frac{1}{ik}\nabla \operatorname{div} J + ikJ - (\nabla \operatorname{div} + k^2)(\eta E) \\
E &= F - g_k * (\nabla \operatorname{div} + k^2)(\eta E)
\end{aligned}$$

2nd kind strongly singular integral equation in $\Omega = \Omega$

$$E - \nabla \operatorname{div} g_k * (\eta E) + k^2 g_k * (\eta E) = F$$

(VIE)

$$E(x) - \nabla_x \int_{\Omega^-} \nabla_y g_k(x-y) \cdot E(y)\eta(y)\,dy + k^2 \int_{\Omega^-} g_k(x-y)E(y)\eta(y)\,dy = F(x)$$

Maxwell $\to$ Helmholtz ("dyadic Green function")

$$-(\Delta + k^2) = \frac{1}{k^2}(\nabla \operatorname{div} + k^2)(\mathbf{curl}\,\mathbf{curl} - k^2)$$

$$\begin{aligned}
\mathbf{curl}\,\mathbf{curl}\, E - k^2 \varepsilon_r E &= ikJ \\
\mathbf{curl}\,\mathbf{curl}\, E - k^2 E &= ikJ - k^2 \eta E \\
-(\Delta + k^2) E &= -\tfrac{1}{ik}\nabla \operatorname{div} J + ikJ - (\nabla \operatorname{div} + k^2)(\eta E) \\
E &= F - g_k * (\nabla \operatorname{div} + k^2)(\eta E)
\end{aligned}$$

2nd kind strongly singular integral equation in $\Omega = \Omega^-$

$$E - \nabla \operatorname{div} g_k * (\eta E) + k^2 g_k * (\eta E) = F$$

(*VIE*)

$$E(x) - \nabla_x \int_{\Omega^-} \nabla_y g_k(x-y) \cdot E(y)\eta(y)\,dy + k^2 \int_{\Omega^-} g_k(x-y) E(y)\eta(y)\,dy = F(x)$$

Thank you

for your attention