The inf-sup constant of the divergence

Martin Costabel

Collaboration with Monique Dauge
with contributions from C. Bernardi, V. Girault, M. Crouzeix, Y. Lafranche

IRMAR, Université de Rennes 1

Mafelap 2013
Brunel University, 11–14 June 2013

The inf-sup Constant: Definition

- Ω bounded domain $\mathbb{R}^d$ ($d \geq 1$). No regularity assumptions.

The inf-sup constant of Ω

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_1 \, \|q\|_0}$$

- $L^2(\Omega)$ space of square integrable functions q on Ω. Norm $\|q\|_0$
- $H^1(\Omega)$ Sobolev space of $v \in L^2(\Omega)$ with gradient $\nabla v \in L^2(\Omega)^d$
- $L^2_\circ(\Omega)$ subspace of $q \in L^2(\Omega)$ with $\int_\Omega q = 0$.
- $H^1_0(\Omega)$ closure in $H^1(\Omega)$ of $C^\infty_0(\Omega)$ (zero trace on $\partial\Omega$)
  (Semi-)Norm $|u|_1 = \|\nabla u\|_0$ equivalent to norm $\|u\|_{H^1(\Omega)}$

$\beta(\Omega)$ is invariant with respect to translations, rotations, dilations.

The inf-sup Constant: A Simple Example

The square $\Omega = (0,1) \times (0,1) =: \square \subset \mathbb{R}^2$

The Square

[Question] : What is $\beta(\square)$?
[Answer] : Unknown !

Conjecture 1 [Horgan-Payne 1983]

$$\beta(\square)^2 = \frac{2}{7} \approx 0.2857... \qquad (\rightarrow\ \beta(\square) \approx 0.5345)$$

C. O. Horgan and L. E. Payne, *On inequalities of Korn, Friedrichs and Babuška-Aziz*. Arch. Rational Mech. Anal., **82** (1983), pp. 165–179.

Conjecture 2 [current]

$$\beta(\square)^2 = \frac{1}{2} - \frac{1}{\pi} \approx 0.18169... \qquad (\rightarrow\ \beta(\square) \approx 0.42625)$$

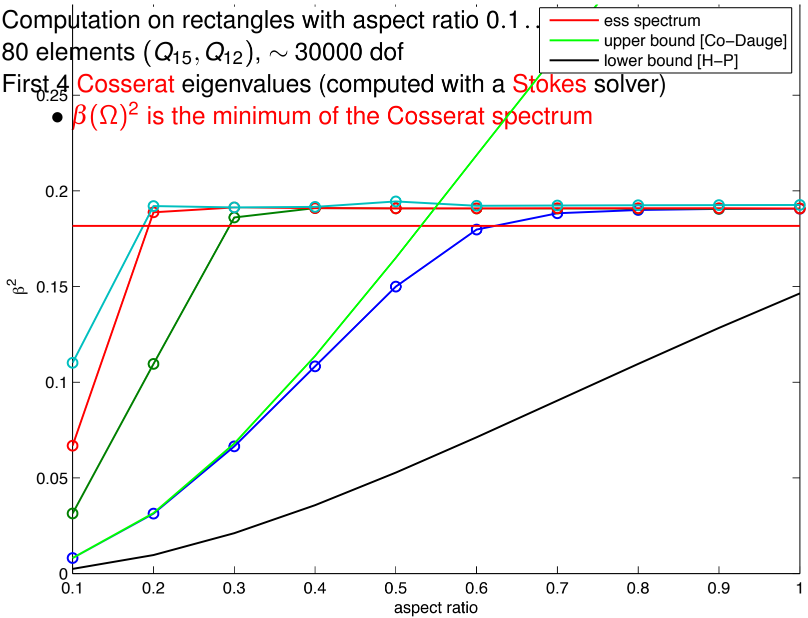
Why not simply compute it ?

Computation on rectangles with aspect ratio 0.1 . .

80 elements (Q_{15}, Q_{12}), $\sim$ 30000 dof

First 4 Cosserat eigenvalues (computed with a Stokes solver)

- $\beta(\Omega)^2$ is the minimum of the Cosserat spectrum



Outline

[0]

Relations 1: Lions' lemma

$H^{-1}(\Omega)$ dual space of $H_0^1(\Omega)$ with dual norm $|\cdot|_{-1}$:
For $q \in L^2_\circ(\Omega)$:

$$|\nabla q|_{-1} = \sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{\langle \nabla q, \boldsymbol{v}\rangle_\Omega}{|\boldsymbol{v}|_1} = \sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_{1,\Omega}}$$

$$\beta(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \frac{|\nabla q|_{-1}}{\|q\|_0}$$

Lemma [Lions 1958, unpublished*, for smooth domains] [Nečas 1967, for Lipschitz domains]

$$\|q\|_0^2 \le C(\Omega)\, |\nabla q|_{-1}^2 \qquad \forall q \in L^2_\circ(\Omega)$$

* According to [E. Magenes and G. Stampacchia 1958].

$$\rightarrow \qquad C(\Omega) = \frac{1}{\beta(\Omega)^2}$$

$$\text{Lions' Lemma} \iff \nabla \quad : L^2_\circ(\Omega) \to H^{-1}(\Omega)^d \text{ is injective with closed range}$$
$$\iff \operatorname{div} \; : H^1_0(\Omega)^d \to L^2_\circ(\Omega) \quad \text{is surjective}$$

Babuška-Aziz inequality [Babuška-Aziz 1971], named by [Horgan-Payne 1983]

$$\Omega \text{ Lipschitz, } q \in L^2_\circ(\Omega) \quad \Longrightarrow \quad \exists\, \boldsymbol{v} \in H^1_0(\Omega)^d : \operatorname{div} \boldsymbol{v} = q$$
$$|\boldsymbol{v}|_1^2 \le C(\Omega) \|q\|_0^2$$

Equivalence for a domain Ω:

$$\beta(\Omega) > 0 \iff \text{Lions' lemma} \iff \text{Babuška-Aziz inequality}$$

This condition (and its discrete counterpart) is called inf-sup condition or LBB condition, after

- **L**adyzhenskaya [?]
- **B**abuška [Babuška 1971-73]
- **B**rezzi [Brezzi 1974]

Lions' Lemma $\iff \nabla \quad : L^2_\circ(\Omega) \to H^{-1}(\Omega)^d$ is injective with closed range

$\iff \operatorname{div} : H^1_0(\Omega)^d \to L^2_\circ(\Omega)$ is surjective

Babuška-Aziz inequality [Babuška-Aziz 1971], named by [Horgan-Payne 1983]

Ω Lipschitz, $q \in L^2_\circ(\Omega) \quad \Longrightarrow \quad \exists\, \boldsymbol{v} \in H^1_0(\Omega)^d : \operatorname{div} \boldsymbol{v} = q$

$$|\boldsymbol{v}|_1^2 \le C(\Omega) \|q\|_0^2$$

Equivalence for a domain Ω:

$$\beta(\Omega) > 0 \iff \text{Lions' lemma} \iff \text{Babuška-Aziz inequality}$$

This condition (and its discrete counterpart) is called inf-sup condition or LBB condition, after

- **L**adyzhenskaya Added by J. T. Oden ca 1980, on suggestion by J.-L. Lions
- **B**abuška [Babuška 1971-73]
- **B**rezzi [Brezzi 1974]

If the LBB condition is satisfied for Ω, Korn's inequality follows:

$$\partial_i\partial_j u_k = \partial_i\varepsilon_{jk} + \partial_j\varepsilon_{jk} - \partial_k\varepsilon_{ij}, \qquad \varepsilon = \tfrac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)$$

$$\Longrightarrow |\nabla\nabla\boldsymbol{u}|_{-1} \sim |\nabla\varepsilon|_{-1} \Longrightarrow \|\nabla\boldsymbol{u}\|_0 \sim \|\varepsilon\|_0$$

Korn's second inequality

If $\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^\top \in L^2_\circ(\Omega)$, then

$$\|\nabla\boldsymbol{u}\|_0^2 \le K(\Omega)\|\varepsilon\|_0^2$$

For $\Omega \subset \mathbb{R}^d$, LBB implies Korn: $K(\Omega) \le 1 + \frac{2(d-1)}{\beta(\Omega)^2}$.

[Horgan-Payne 1983]

For $\Omega \subset \mathbb{R}^2$, Ω smooth: $K(\Omega) = 2C(\Omega) = \frac{2}{\beta(\Omega)^2}$.

Open problem

Does this hold without smoothness condition?

[E.&F. Cosserat 1898]

Find $\boldsymbol{u} \in H_0^1(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$ such that

$$\sigma \Delta \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u} = 0\,.$$

Aim: Solving the Lamé Dirichlet problem by eigenfunction expansion.

Equivalent eigenvalue problems:

$$\Delta^{-1} \nabla \operatorname{div} \boldsymbol{u} = \sigma \boldsymbol{u} \qquad \text{in } H_0^1(\Omega)^d$$

or, for $\sigma \neq 0$:

$$\operatorname{div} \Delta^{-1} \nabla q = \sigma q \qquad \text{in } L^2_\circ(\Omega).$$

Definition: Cosserat operator $\boxed{\mathscr{S} = \operatorname{div} \Delta^{-1} \nabla}$ Selfadjoint, positive, ≤ 1.

This is not an elliptic eigenvalue problem! $\sigma = 1$ has infinite multiplicity

$q = \Delta \phi$, $\phi \in H_0^2(\Omega) \Rightarrow \Delta^{-1} \nabla q = \nabla \phi \Rightarrow \mathscr{S} q = q$.

Cosserat eigenfunctions [E.&F. Cosserat 1898]

For ellipsoids, the Cosserat eigenvalue problem can be solved explicitly.
If Ω is the unit ball, one has: Let q be a harmonic polynomial, homogeneous of degree k, then

$$\mathscr{S}q = \sigma_k q \qquad \text{with} \quad \sigma_k = \left(2 + \tfrac{d-2}{k}\right)^{-1}$$

Define $\boxed{\sigma(\Omega) = \min(\text{ Spectrum } \mathscr{S})}$ Ball in $\mathbb{R}^d$: $\sigma(\Omega) = \frac{1}{d}$.

A simple relation

$$\sigma(\Omega) = \beta(\Omega)^2.$$

Proof: $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Riesz isometry. Let $q \in L^2_\circ(\Omega)$

$$\big\langle Sq, q\big\rangle_\Omega = \big\langle \operatorname{div} \Delta^{-1} \nabla q, q\big\rangle_\Omega = \big\langle \nabla q, -\Delta^{-1}\nabla q\big\rangle_{H^{-1}(\Omega)^d, H_0^1(\Omega)^d} = |\nabla q|^2_{-1}$$

$$\sigma(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \frac{\big\langle Sq, q\big\rangle_\Omega}{\big\langle q, q\big\rangle_\Omega} = \inf_{q \in L^2_\circ(\Omega)} \frac{|\nabla q|^2_{-1}}{\|q\|^2_0} = \frac{1}{C(\Omega)} = \beta(\Omega)^2$$

The Stokes system

Find $\boldsymbol{u} \in H_0^1(\Omega)$, $p \in L^2_\circ(\Omega)$:

$$-\Delta \boldsymbol{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega$$
$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega$$

The Cosserat operator $\mathscr{S}$ is the Schur complement of the Stokes system:
The pressure p satisfies the equation

$$\mathscr{S} p = \operatorname{div} \Delta^{-1} f.$$

The Cosserat operator is the error reduction operator in Uzawa's iterative algorithm...

Relations 5: Cosserat and Stokes

The Cosserat eigenvalue problem is a Stokes eigenvalue problem

Find $\boldsymbol{u} \in H^1_0(\Omega)$, $p \in L^2_\circ(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= 0 \qquad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= \sigma p \qquad \text{in } \Omega \end{aligned}$$

An exercise:

For fixed $q \in L^2_\circ(\Omega)$, the $\sup\limits_{\boldsymbol{v}\in H^1_0(\Omega)^d} \dfrac{\langle \operatorname{div} \boldsymbol{v}, q\rangle}{|\boldsymbol{v}|_1}$ is attained for $\boldsymbol{v} = \boldsymbol{v}^{(0)} = \Delta^{-1}\nabla q$.

$$\boxed{\beta(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \frac{1}{\|q\|_0} \frac{\langle \operatorname{div} \boldsymbol{v}^{(0)}(q), q\rangle}{|\boldsymbol{v}^{(0)}(q)|_1}}$$

On the other hand, for $\boldsymbol{v} = \boldsymbol{v}^{(1)} \in H^1_0(\Omega)^d$, solution of

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla p &= 0 \qquad \text{in } \Omega \\ \operatorname{div} \boldsymbol{v} &= q \qquad \text{in } \Omega \end{aligned}$$

there holds

$$\boxed{\beta(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \frac{1}{\|q\|_0} \frac{\langle \operatorname{div} \boldsymbol{v}^{(1)}(q), q\rangle}{|\boldsymbol{v}^{(1)}(q)|_1}}$$

Show that for $q \in L^2_\circ(\Omega)$

$$\frac{\langle \operatorname{div} \boldsymbol{v}^{(0)}(q), q\rangle}{|\boldsymbol{v}^{(0)}(q)|_1} = \frac{\langle \operatorname{div} \boldsymbol{v}^{(1)}(q), q\rangle}{|\boldsymbol{v}^{(1)}(q)|_1}$$

if and only if q is a Cosserat eigenfunction.

For $\sigma \notin \{0, \frac{1}{2}, 1\}$, the operator $A_\sigma = -\sigma\Delta + \nabla\,\mathrm{div}$ is elliptic.
If $\Omega \subset \mathbb{R}^2$ has a corner of opening ω, one can therefore determine the corner singularities via Kondrat'ev's method of Mellin transformation:
Look for solutions of the form $r^\lambda \phi(\theta)$ in a sector. $\to q \sim r^{\lambda-1}\phi(\theta)$
Characteristic equation (Lamé system, known!) for a corner of opening ω:

$$(*) \qquad (1-2\sigma)\frac{\sin\lambda\omega}{\lambda} = \pm\sin\omega.$$

Theorem [Kondrat'ev 1967]

For $\sigma \in [0,1] \setminus \{0, \frac{1}{2}, 1\}$, $A_\sigma : H_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is Fredholm iff the equation $(*)$ has no solution on the line $\Re e\,\lambda = 0$.

With $z = \lambda\omega$, we rewrite $(*)$:

$$(1-2\sigma)\frac{\sin z}{z} = \pm\frac{\sin\omega}{\omega}.$$

Result :

- $(*)$ has roots on the line $\Re e\,\lambda = 0$ iff $|1-2\sigma|\omega \le |\sin\omega|$
- If $|1-2\sigma|\omega > |\sin\omega|$, there is a root $\lambda \in (0,1)$

Theorem [Co & Dauge ca 2000]

$\Omega \subset \mathbb{R}^2$ piecewise smooth with corners of opening ω_j.

$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}) = \bigcup_{\text{corners } j} \left[\tfrac{1}{2} - \frac{|\sin\omega_j|}{2\omega_j}, \tfrac{1}{2} + \frac{|\sin\omega_j|}{2\omega_j}\right] \cup \{1\}$$

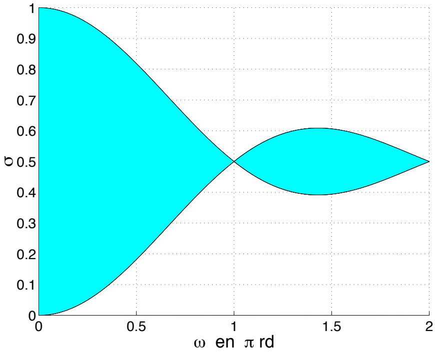


Figure: Essential spectrum: σ vs. opening ω

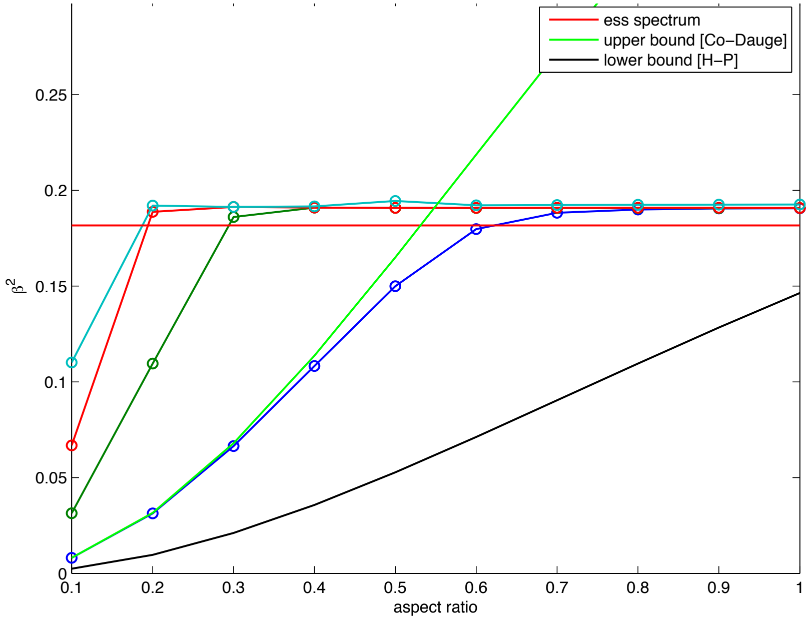
Example : Rectangle, $\omega = \frac{\pi}{2}$

$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}\big|_M) = [\tfrac{1}{2} - \tfrac{1}{\pi}, \tfrac{1}{2} + \tfrac{1}{\pi}] \approx [0.181, 0.818]$$

Corollary

$$\beta(\square)^2 \leq \tfrac{1}{2} - \tfrac{1}{\pi}$$

Rectangle: First 4 eigenfunctions



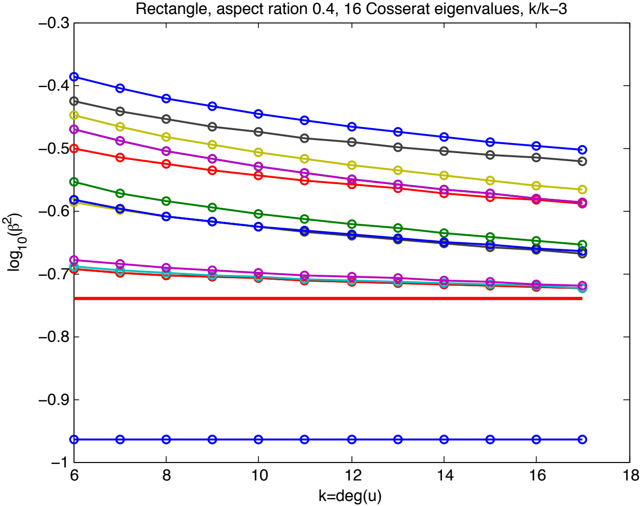
Rectangle: Convergence of first 2 eigenvalues, *p* version

Rectangle, aspect ratio 0.4
First 2 Cosserat eigenvalues

degree	ev 1	ev 2
6, 3	0.108330171273605	0.202677364134760
7, 4	0.108328806867517	0.200312078158581
8, 5	0.108328239442393	0.198542331544704
9, 6	0.108327944346408	0.197116814521769
10, 7	0.108327777628900	0.195905723376609
11, 8	0.108327678000311	0.194831095631923
12, 9	0.108327616075105	0.193840557639462
13, 10	0.108327576463480	0.192895071533516
14, 11	0.108327550603459	0.191962461904639
15, 12	0.108327533495880	0.191013598891857
16, 13	0.108327522104368	0.190020022260760
17, 14	0.108327514523012	0.188952593727730

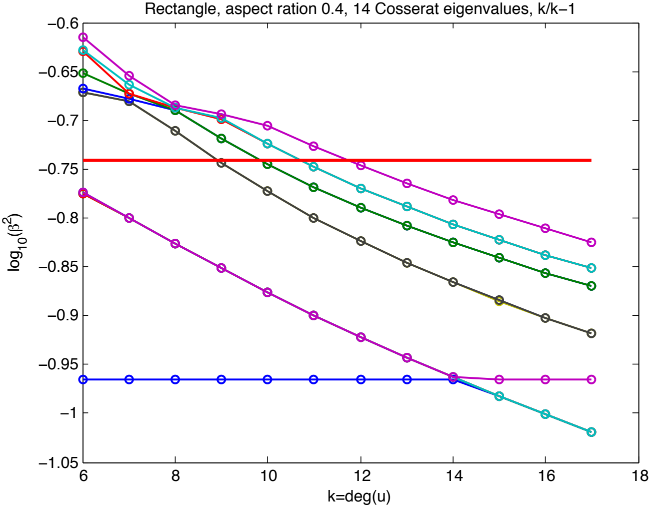
Rectangle, aspect ratio 0.4

First 13 Cosserat eigenvalues, (Q_k, Q_{k-3})



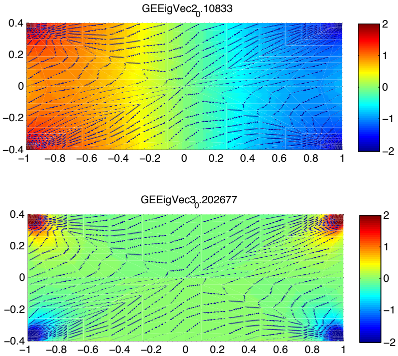
Rectangle, aspect ratio 0.4

First 13 Cosserat eigenvalues, (Q_k, Q_{k-1}) “Taylor-Hood”

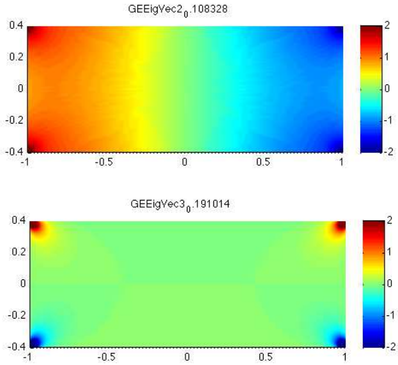


Rectangle, aspect ratio 0.4

Degrees: $6, 3$

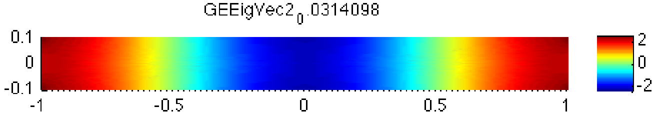


Degrees: 15,12



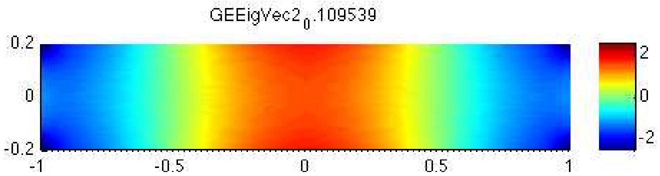
Rectangle: First eigenfunction

Rectangle, aspect ratio 0.1



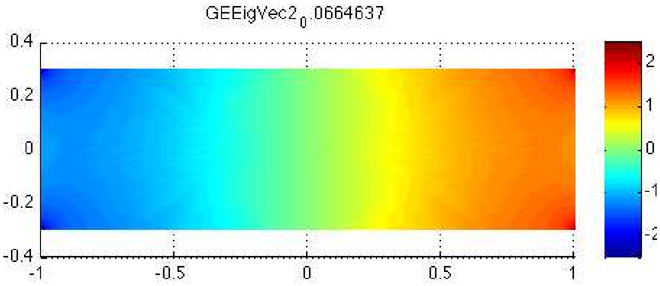
Rectangle: First eigenfunction

Rectangle, aspect ratio 0.2



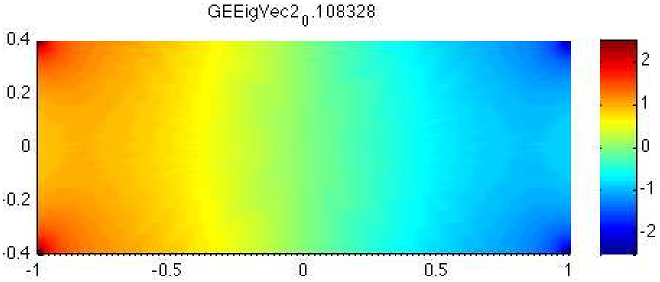
Rectangle: First eigenfunction

Rectangle, aspect ratio 0.3



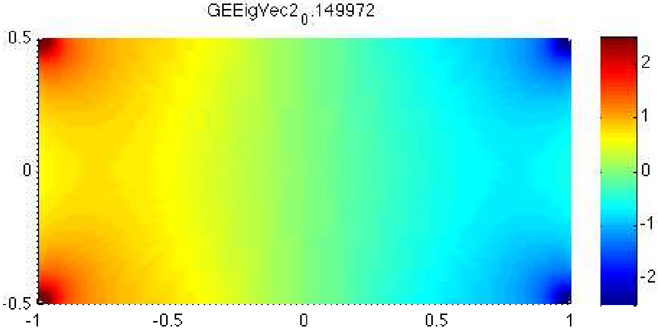
Rectangle: First eigenfunction

Rectangle, aspect ratio 0.4

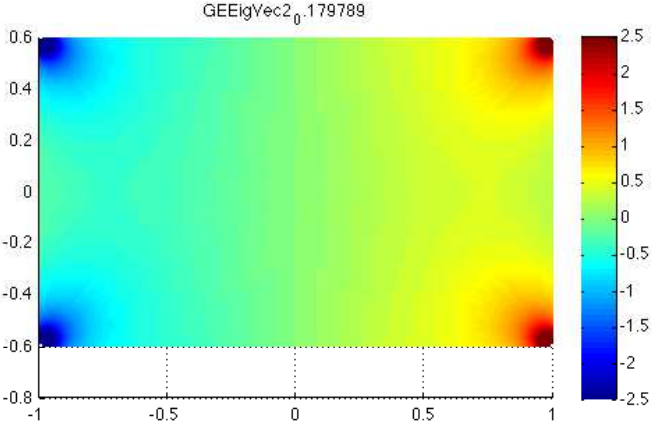


Rectangle: First eigenfunction

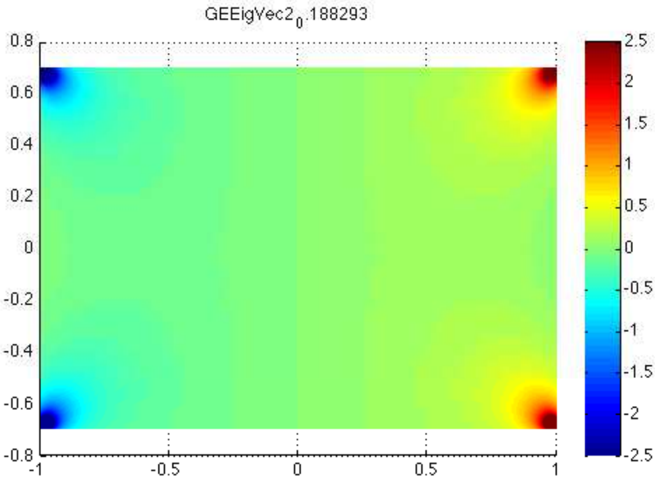
Rectangle, aspect ratio 0.5



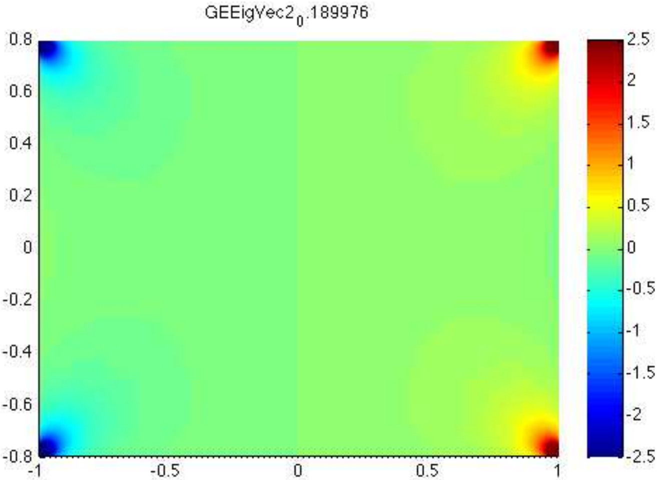
Rectangle, aspect ratio 0.6



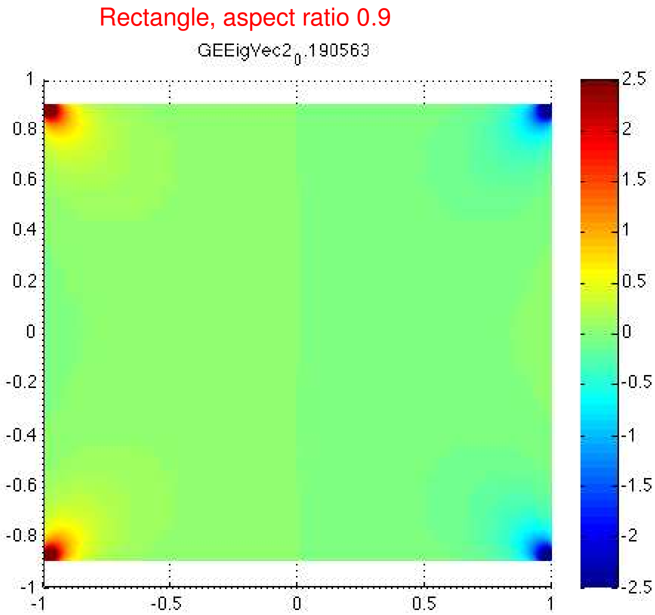
Rectangle, aspect ratio 0.7

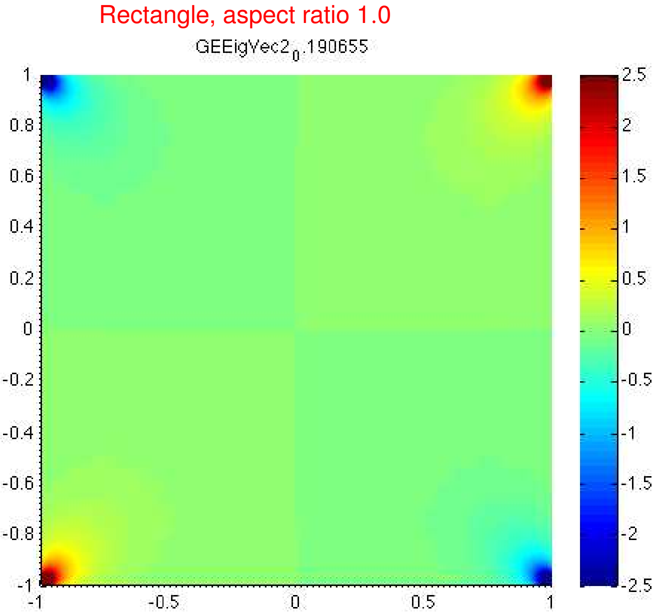


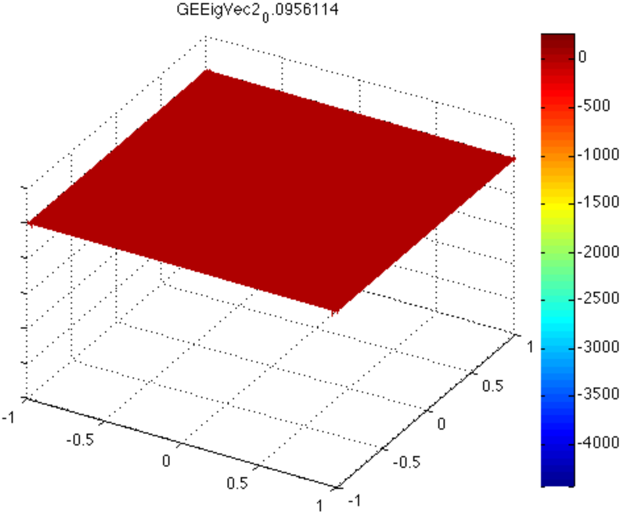
Rectangle, aspect ratio 0.8



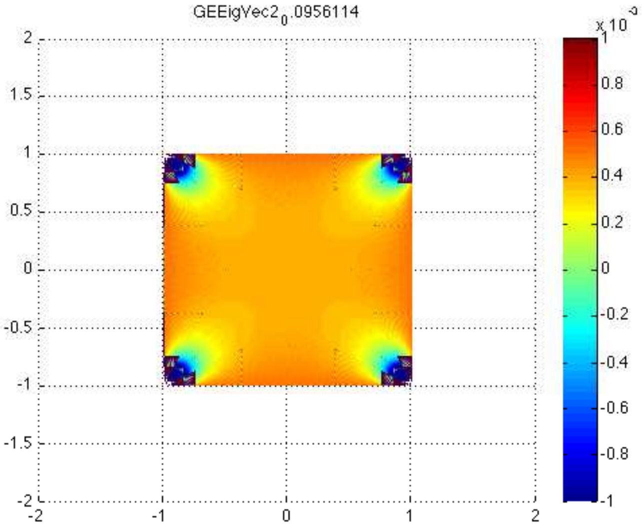
Rectangle: First eigenfunction



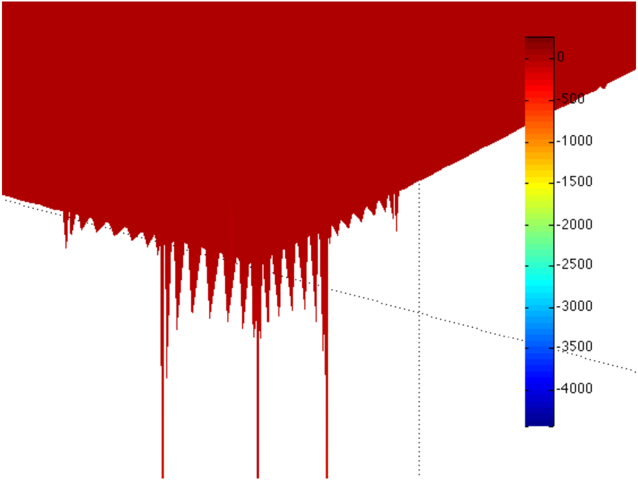

Rectangle, aspect ratio 1.0
GEEigVec2_0.190655
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5


GEEigVec2_0.0956114
0
-500
-1000
-1500
-2000
-2500
-3000
-3500
-4000
-1
-0.5
0
0.5
1
-1
-0.5
0
0.5
1

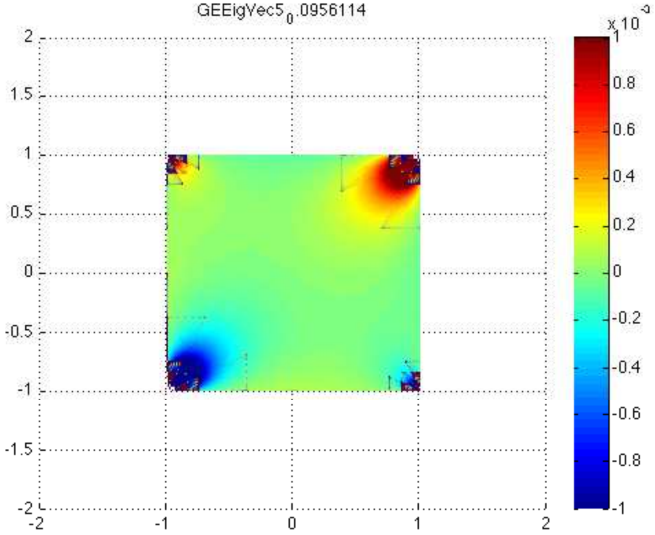
Square: First eigenfunction, (Q_{17}, Q_{16})



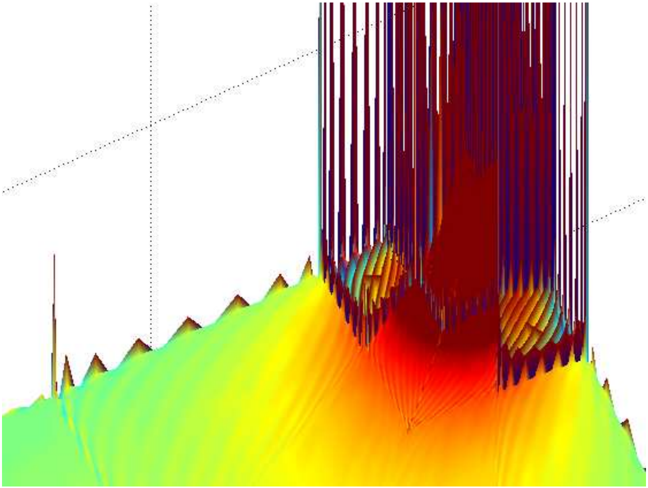
Square: First eigenfunction, (Q_{17}, Q_{16})



Square: Fourth eigenfunction, (Q_{17}, Q_{16})



Square: Fourth eigenfunction, (Q_{17}, Q_{16})



Upper and lower bounds

Let $\Omega \subset \mathbb{R}^d$ be starshaped with respect to a ball B and $\omega \in C_0^\infty(B)$ be such that $\int \omega = 1$.

Define $\quad \mathbf{T}p(x) = \int_\Omega \mathbf{G}(x,y)p(y)\,dy \quad$ with

$$\mathbf{G}(x,y) = \frac{\boldsymbol{x}-\boldsymbol{y}}{|x-y|^d} \int_{|x-y|}^\infty \omega\Big(y + t\frac{x-y}{|x-y|}\Big)\, t^{d-1}\,dt$$

Then $\mathbf{T} : L^2_\circ(\Omega) \to H^1_0(\Omega)^d$ is continuous and $\operatorname{div}\mathbf{T}p = p$ (right inverse!).

Explanation :

The adjoint operator T' is the regularized Poincaré path integral

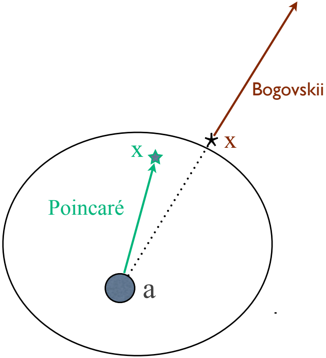
$$\mathrm{T}'\boldsymbol{u}(x) = \int_B \omega(a)\int_a^x \boldsymbol{u}\cdot\mathbf{ds}\,da = \int_B \omega(a)(\boldsymbol{x}-\boldsymbol{a})\cdot\int_0^1 \boldsymbol{u}\big(a + t(x-a)\big)\,dt\,da$$

satisfying $\mathrm{T}'\nabla p(x) = p(x) - \int_B p(a)\omega(a)\,da$ (left inverse on $L^2(\Omega)/\mathbb{R}$)

Lemma [Co&McIntosh 2010]

$\mathbf{T}$ and T' are pseudodifferential operators on $\mathbb{R}^d$ of order -1.

$\forall s \in \mathbb{R}$: $\quad \mathbf{T} : \widetilde{H}^s(\Omega) \to \widetilde{\boldsymbol{H}}^{s+1}(\Omega)$ and $\mathrm{T}' : \boldsymbol{H}^s(\Omega) \to H^{s+1}(\Omega)$



Support properties:

- For $x \in \Omega$, $\mathrm{T}'\boldsymbol{u}(x)$ depends only on $\boldsymbol{u}\big|_{\Omega}$
- If $p = 0$ on $\mathbb{R}^d \setminus \Omega$, then $\mathbf{T}p = 0$ on $\mathbb{R}^d \setminus \Omega$.

Starshaped domains

Theorem [Bogovskiĭ 1979], [Galdi 1994]

Let $\Omega \subset \mathbb{R}^n$ be contained in a ball of radius R, starshaped with respect to a concentric ball of radius ρ. There exists a constant γ_d only depending on the dimension d such that

$$\beta(\Omega) \geq \gamma_d \left(\frac{\rho}{R}\right)^{d+1}$$

Corollary

Let Ω be a finite union of bounded starshaped domains.

Then $\beta(\Omega) > 0$.

This includes all bounded Lipschitz domains, possibly with cracks.

In dimension $d = 2$, we can prove

$$\beta(\Omega) \geq \frac{\rho}{2R}$$

M. Costabel, M. Dauge: On the inequalities of Babuška-Aziz, Friedrichs and Horgan-Payne. arXiv 2013.

John domains

Theorem [Acosta-Durán-Muschietti 2006], [Durán 2012]

Let $\Omega \subset \mathbb{R}^d$ be a bounded John domain. Then $\beta(\Omega) > 0$.

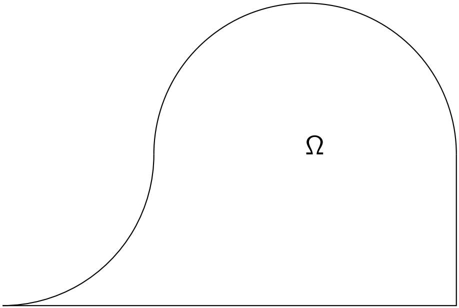


Figure: Not a John domain: Outward cusp, $\beta(\Omega) = 0$ [Friedrichs 1937]

Definition

A domain $\Omega \subset \mathbb{R}^d$ with a distinguished point $\boldsymbol{x}_0$ is called a John domain if it satisfies the following "twisted cone" condition:
There exists a constant $\delta > 0$ such that, for any $\boldsymbol{y}$ in Ω, there is a rectifiable curve γ: $[0,\ell] \to \Omega$ parametrized by arclength such that

$$\gamma(0) = \boldsymbol{y}, \quad \gamma(\ell) = \boldsymbol{x}_0, \quad \text{and} \quad \forall t \in [0,\ell] : \ \operatorname{dist}(\gamma(t), \partial\Omega) \geq \delta\, t\,.$$

Here $\operatorname{dist}(\gamma(t), \partial\Omega)$ denotes the distance of $\gamma(t)$ to the boundary $\partial\Omega$.

Example : Every weakly Lipschitz domain is a John domain.

A John domain: Union of Lipschitz domains

A John domain: Zigzag

Figure: A weakly Lipschitz domain: the self-similar zigzag

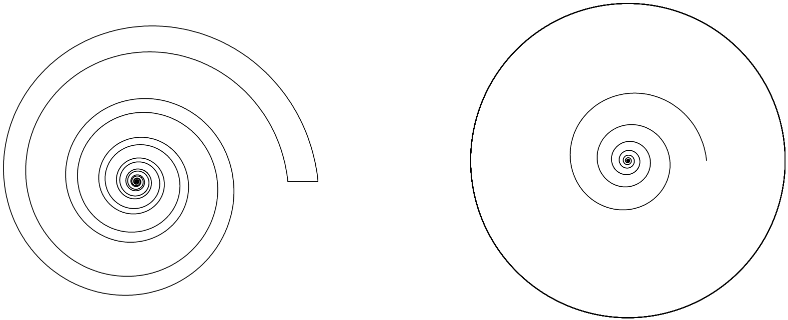


Figure: Weakly Lipschitz (left), John domain (right)

Fractal John domains: Tree or Lung

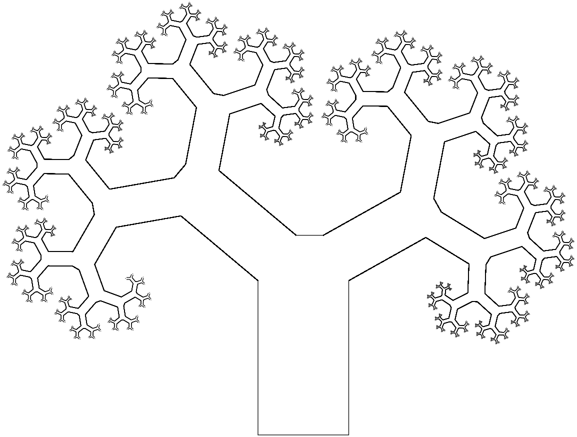


Figure: A John domain: the infinite tree

Friedrichs' inequality [named by Horgan-Payne 1983]

Let $\Omega \subset \mathbb{R}^2$ be a bounded piecewise smooth domain with no outward cusps. There exists a constant $\Gamma(\Omega)$ such that for any holomorphic $f + ig \in L^2_\circ(\Omega)$ there holds

$$\|f\|_0^2 \le \Gamma(\Omega)\|g\|_0^2$$

Theorem [H-P 1983 for $\Omega \in C^2$], [Co-Dauge 2013 without smoothness condition]

For any bounded domain $\Omega \subset \mathbb{R}^2$

$$\frac{1}{\beta(\Omega)^2} = \Gamma(\Omega) + 1\,.$$

Sketch of Proof:

If $\beta(\Omega) > 0$ and $f + ig \in L^2_\circ(\Omega)$ holomorphic, then one uses the Babuška-Aziz inequality and

$$\langle f, \operatorname{div} \boldsymbol{u} \rangle_\Omega = -\langle g, \operatorname{curl} \boldsymbol{u} \rangle_\Omega$$

to show that $\|f\|_0^2 \le (C(\Omega) - 1)\|g\|_0^2$.

Conversely, if $p \in L^2_\circ(\Omega)$ is given, define

$$\boldsymbol{u} = \Delta^{-1}\nabla p, \quad q = \operatorname{div} \boldsymbol{u} \quad \text{and } g = \operatorname{curl} \boldsymbol{u}$$

Then one can see that g and $q - p$ are conjugate harmonic functions in $L^2_\circ(\Omega)$ and that the Friedrichs inequality implies

$$\|p\|_0^2 \le (\Gamma(\Omega) + 1)\,|\nabla p|_{-1}^2 .$$

Let $\Omega \subset \mathbb{R}^2$ be star-shaped with respect to a ball. Boundary in polar coordinates

$$r = f(\theta) \qquad \text{with } f \text{ Lipschitz}\,, \max_{\theta\in[0,2\pi]} f(\theta) = 1\,.$$

Define the angle $\gamma(\theta)$ between the radius vector and the normal vector

$$\tan\gamma(\theta) = \frac{f'(\theta)}{f(\theta)}$$

Set

$$P(\alpha,\theta) = \frac{1}{\alpha f(\theta)^2}\left(1 + \frac{\tan^2\gamma(\theta)}{1-\alpha f(\theta)^2}\right)$$

$$M(\Omega) := \inf_{\alpha\in(0,1)}\left\{\sup_{\theta\in[0,2\pi]} P(\alpha,\theta)\right\}; \quad m(\Omega) = \sup_{\theta\in[0,2\pi]}\left\{\inf_{\alpha\in\left(0,\frac{1}{f(\theta)^2}\right)} P(\alpha,\theta)\right\}$$

$$P(\alpha,\theta) = \frac{1}{\alpha f(\theta)^2}\left(1 + \frac{\tan^2\gamma(\theta)}{1-\alpha f(\theta)^2}\right)$$

$$M(\Omega) := \inf_{\alpha\in(0,1)}\left\{\sup_{\theta\in[0,2\pi]} P(\alpha,\theta)\right\}; \qquad m(\Omega) = \sup_{\theta\in[0,2\pi]}\left\{\inf_{\alpha\in\left(0,\frac{1}{f(\theta)^2}\right)} P(\alpha,\theta)\right\}$$

The Horgan-Payne inequality [H-P 1983]

[HPI] $$\Gamma(\Omega) \le m(\Omega)$$

Theorem [H-P 1983]

$$\Gamma(\Omega) \le M(\Omega)$$

The Horgan-Payne inequality

Definition: The Horgan-Payne angle [Stoyan 2001]

$$\omega(\Omega) = \frac{\pi}{2} - \max|\gamma(\theta)|$$

Minimal angle between radius vector and tangent.

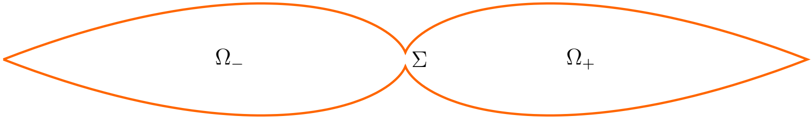
$$[\text{HPI}] \iff \Gamma(\Omega) \le m(\Omega) \iff \boxed{\beta(\Omega) \ge \sin\frac{\omega(\Omega)}{2}}$$

Theorem [Co-Dauge 2013]

1. For circles, ellipses, polygons that have a circumscribed or inscribed circle, hence for all triangles, rectangles, regular polygons:

$$m(\Omega) = M(\Omega)$$

2. There exist domains for which the Horgan-Payne inequality does not hold.



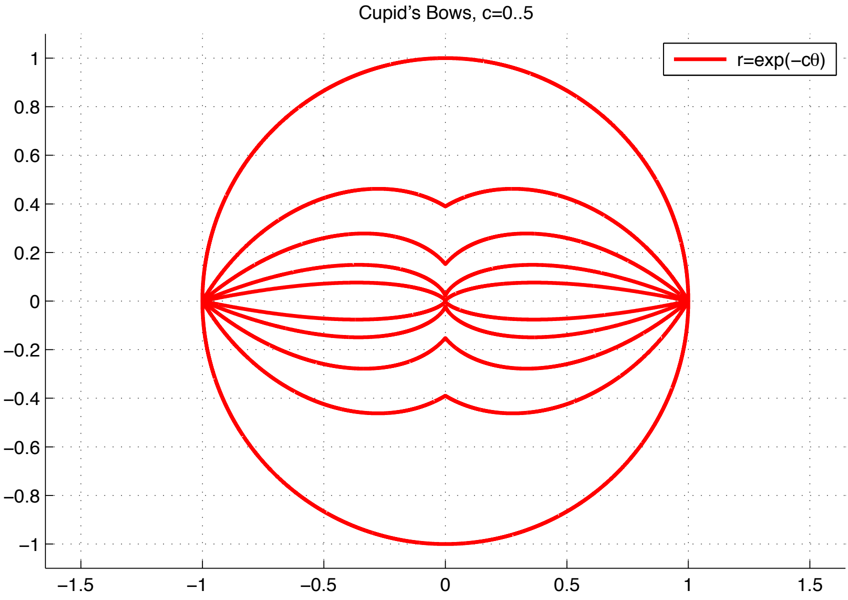
Theorem [Co-Dauge 2013]

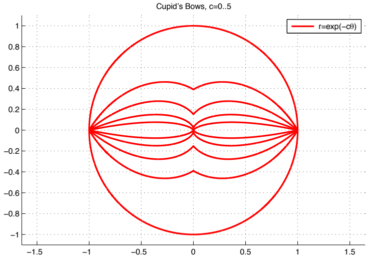
Let $\Omega \in \mathbb{R}^2$ be a disjoint union of Ω_-, Ω_+ and a segment Σ of length L. Then

$$\beta(\Omega)^2 \leq \frac{8}{3} \frac{|\Omega|}{|\Omega_+||\Omega_-|} L^2 .$$

The constant $\frac{8}{3}$ can probably be improved to $\frac{\pi}{16}$ [with M. Crouzeix].

Cupid's Bow





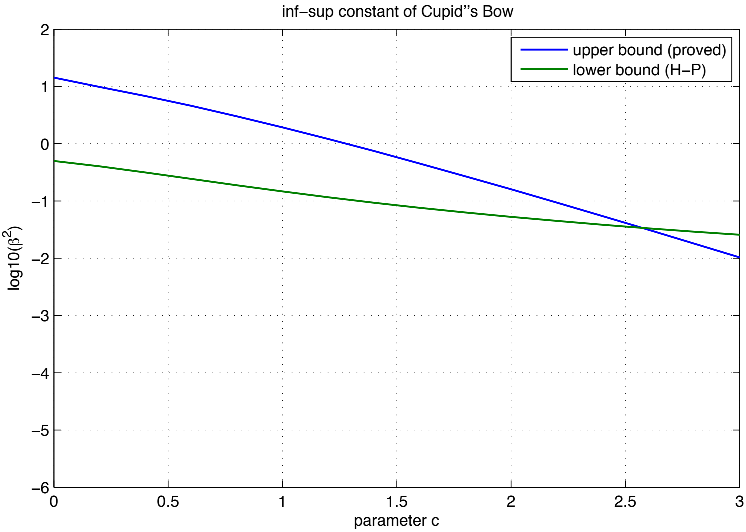
Horgan-Payne angle: $\omega(\Omega) = \arctan \frac{1}{c}$

Horgan-Payne inequality:

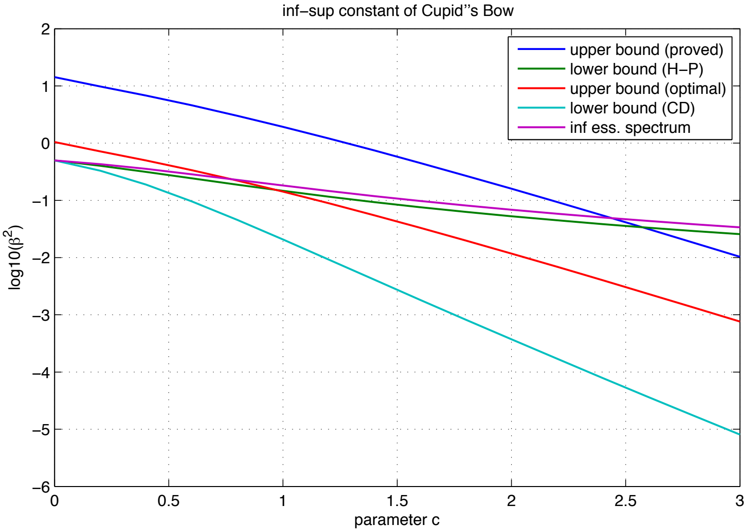
$$\beta(\Omega)^2 \geq \frac{\sqrt{c^2+1} - c}{2\sqrt{c^2+1}} = \frac{1}{4c^2} + O(c^{-4}) \quad \text{as } c \to \infty .$$

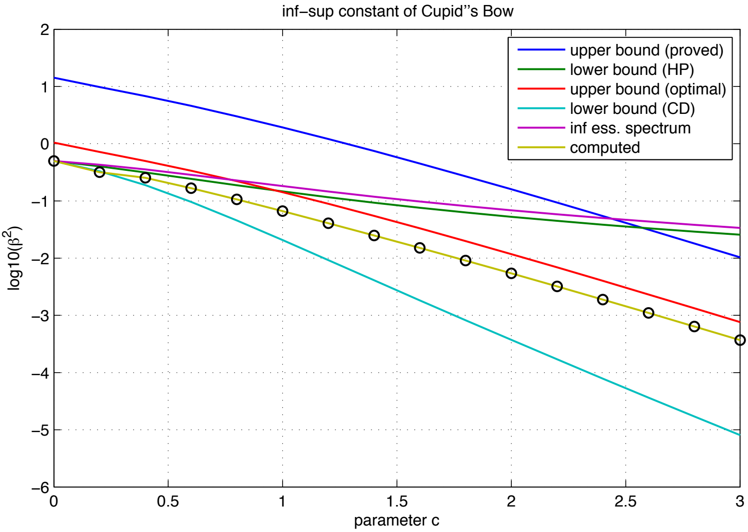
Upper bound

$$\beta(\Omega)^2 \leq \frac{128}{3} \frac{c\, e^{-c\pi}}{1 - e^{-c\pi}} . \qquad [\frac{128}{3} \to \pi]$$


inf−sup constant of Cupid''s Bow
upper bound (proved)
lower bound (H−P)
log10(β2)
parameter c
2
1
0
−1
−2
−3
−4
−5
−6
0
0.5
1
1.5
2
2.5
3

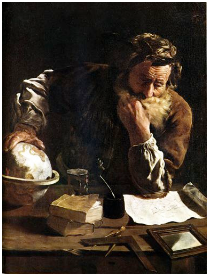
Cupid's Bow


inf–sup constant of Cupid''s Bow
upper bound (proved)
lower bound (H–P)
upper bound (optimal)
lower bound (CD)
inf ess. spectrum
log10(β²)
parameter c
2
1
0
−1
−2
−3
−4
−5
−6
0
0.5
1
1.5
2
2.5
3


inf−sup constant of Cupid''s Bow
upper bound (proved)
lower bound (HP)
upper bound (optimal)
lower bound (CD)
inf ess. spectrum
computed
log10(β²)
parameter c
2
1
0
−1
−2
−3
−4
−5
−6
0
0.5
1
1.5
2
2.5
3

Open Problems

1. Is $\beta(\Box)^2 = \frac{1}{2} - \frac{1}{\pi}$?
2. How to compute $\beta(\Omega)$ reliably? Special elements?
3. Stability of $\beta(\Omega)$ with respect to perturbations of the domain.
   OK for C^2 perturbations. What about $W^{1,\infty}$ perturbations?
4. Equivalence with Korn's inequality.



Thank you for your attention!