3 $\times$ 60 Years of Integral Equation Methods

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Waves Diffracted
by
Patrick Joly

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Present Days

P. JOLY, J. RODRÍGUEZ:
Mathematical aspects of variational boundary integral equations for time dependent wave propagation.
J. Integral Equations Appl. 29 (2017), no. 1, 137–187.

Historical time frame: 3×60 years

- 1838–1840 Gauss: 2 papers, 1 book on Magnetism, Potential Theory
  Single layer potential, 1st kind integral equation, computations

- 1896 Poincaré: “La méthode de Neumann et le problème de Dirichlet”

- 1956–1957 Calderón – Zygmund: “On singular integrals”
  1959–1964 Agmon – Douglis – Nirenberg: “Estimates near the boundary. . . ”

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1870–1877 C. Neumann: Double layer potential, 2nd kind integral equation

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1900–1903 Fredholm: "Sur une nouvelle méthode pour la résolution
du problème de Dirichlet."

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1965 Pseudodifferential Operators

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•1896	Poincaré: “La méthode de Neumann et le problème de Dirichlet”
1899–1904	Korn, Stekloff, Zaremba
1904	Plemelj
1900–1903	Fredholm: “Sur une nouvelle méthode pour la résolution du problème de Dirichlet.”
•1956–1957	Calderón – Zygmund: “On singular integrals”
1959–1964	Agmon – Douglis – Nirenberg: “Estimates near the boundary. . . ”
1965	Pseudodifferential Operators
1973	Nedelec – Planchard: “Une méthode variationnelle d’éléments finis pour la résolution numérique d’un problème extérieur dans $\mathbb{R}^3$”
1976	Wendland – Hsiao et. al.: Boundary element methods

3 Stories

1. “The curious case of Gauss’ missing theorem”.
Or “The single layer potential from 1839 to 1973”.

2. “Convergence of Neumann’s series”.
Or “The norm of the double layer potential from 1870 to 2007”.

3. “Symmetry of the spectrum of the double layer potential in 2D”.
Or “Interplay between boundary integral equations, volume integral equations, and variational methods”.

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Some Recent Papers

- O. Steinbach, W. L. Wendland:
  On C. Neumann's method for second-order elliptic systems in domains with non-smooth boundaries.
  J. Math. Anal. Appl. 262 (2001) 733–748.
- M. Costabel:
  Some historical remarks on the positivity of boundary integral operators.
  In *Boundary element analysis*, volume 29 of *Lect. Notes Appl. Comput. Mech.*, pages 1–27. Springer, Berlin, 2007.
- D. Khavinson, M. Putinar, H. Shapiro:
  On Poincaré's variational problem in potential theory.
  Arch. Ration. Mech. Anal. 185 (2007), 143-184.
- M. Costabel, E. Darrigrand, H. Sakly:
  Volume integral equations for electromagnetic scattering in two dimensions.
  Computers and Mathematics with Applications 70(8) (2015), 2087-2101.

Ω bounded Lipschitz domain in $\mathbb{R}^n$, $\Gamma = \partial\Omega$.
Gauss 1839: First kind integral equation for the gravity potential ($n = 3$)

$$V\phi(x) \equiv \int_\Gamma \frac{\phi(y)\,ds(y)}{4\pi|x-y|} = f(x), \quad x \in \Gamma$$

Variational approach: Minimize $\frac{1}{2}\langle \phi, V\phi\rangle - \langle f, \phi\rangle$.
Needed: Bilinear form $\langle \phi, V\psi\rangle$ is positiv definite.

- 2 principal methods: With or without looking at the integral operator.

Looking at the kernel: obvious estimate

$$\int \frac{\phi(x)\phi(y)}{|x-y|} ds(y) ds(x) \ge \frac{\|\phi\|^2_{L^1(\Gamma)}}{\mathrm{diam}(\Gamma)} \quad \text{if } \phi \ge 0$$

Gauss himself deplored that he needed the positivity of ϕ ($\to$ variational inequality) and wished that one could prove positivity of the bilinear form without this assumption, but found that this is "not evident".
Mystery: He had (almost) all the ingredients in his paper:
Jump relations and Green's formula.

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Jump relation for the single layer potential ($\Omega^- = \Omega$, $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}$):

$$u(x) = \mathscr{S}\phi(x) = \int_\Gamma \frac{\phi(y)\,ds(y)}{4\pi|x-y|} \text{ in } \mathbb{R}^3 \qquad \Longrightarrow \qquad \phi = -[\partial_n u]_\Gamma = \partial_n^- u - \partial_n^+ u$$

Green's formula

$$\Delta u = 0 \quad \Longrightarrow \quad \int_{\Omega^\pm} |\nabla u|^2\, dx = \mp \int_\Gamma u \partial_n^\pm u\, ds$$

Adding up ($V\phi = u|_\Gamma$, $\phi = -[\partial_n u]_\Gamma$):

$$\boxed{\langle \phi, V\phi \rangle = \int_{\mathbb{R}^3} |\nabla \mathscr{S}\phi|^2\, dx}$$

This is > 0 if $\phi \neq 0$.

$\|\phi\|_V^2 = \langle \phi, V\phi \rangle$ defines a norm on $H^{-\frac{1}{2}}(\Gamma)$, equivalent to the Sobolev norm.

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Story 1: Gauss' Missing Theorem, proved (without looking at the kernel)

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[Nedelec-Planchard 1973]

$\|\phi\|_V^2 = \langle \phi, V\phi \rangle$ defines a norm on $H^{-\frac{1}{2}}(\Gamma)$, equivalent to the Sobolev norm.

$$\mathscr{D}v(x) = \frac{1}{4\pi}\int_\Gamma v(y)\partial_{n(y)}|x-y|^{-1}ds(y)\,, \quad Kv = \mathscr{D}v|_\Gamma$$

Jump relations for the double layer potential $u = \mathscr{D}v$

$$[\partial_n u]_\Gamma = 0\,; \quad [\gamma u]_\Gamma = v\,; \quad \gamma^\pm u = (\pm\tfrac{1}{2} + K)v$$

2nd kind integral equation for the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on Γ

$$(\tfrac{1}{2} - K)v = -g \quad \text{or} \quad (1 - N)v = -2g \quad \text{with } N = 2K$$

If one can show that N is a contraction in some Banach space, one gets a unique solution by successive approximation ("Neumann series")

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First approach (looking at the kernel)

$$d\theta_x(y) = -\frac{n(y)\cdot(y-x)}{2\pi|x-y|^3}ds(y)$$

is for $x \in \Gamma$ a measure (solid angle) of total mass 1 on Γ,
positive if Ω is convex.

[C. Neumann 1877] Using hard analysis

If Ω is convex, but not the intersection of 2 convex cones, then $N = 2K$ is a contraction on $L^\infty(\Gamma)/\mathbb{R}$ in a norm equivalent to the L^∞ norm.

Second approach (without integral operators)

[Poincaré 1896] An energy inequality

There exists a constant $\mu > 0$ depending on Ω such that

1. If u is a double layer potential, then
$$\frac{1}{\mu}\int_{\Omega^+}|\nabla u|^2 \le \int_{\Omega}|\nabla u|^2 \le \mu\int_{\Omega^+}|\nabla u|^2$$
2. If u is a single layer potential, then
$$\int_{\Omega}|\nabla u|^2 \le \mu\int_{\Omega^+}|\nabla u|^2 \quad \text{and if } \int_{\Gamma} u = 0 \text{ then } \int_{\Omega^+}|\nabla u|^2 \le \mu\int_{\Omega}|\nabla u|^2$$

Poincaré: Proved for simply connected smooth domains.
Korn, Stekloff. . . : For Lyapunov domains.
Nowadays easy exercise for Lipschitz domains (Ω^+ connected).

[Steklov 1900]

Nous appellerons ce théorème *théorème fondamental*. . . .
Nous verrons dans ce qui va suivre, que *la solution de tous les problèmes fondamentaux de la Physique mathématique se ramène à la démonstration complète du théorème fondamental.*

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Story 2: Norm of the double layer potential, Poincaré et. al.

$$u = \mathscr{S}\phi \quad \Longrightarrow \quad \int_{\Omega^\pm} |\nabla u|^2 = \mp \int_\Gamma u \partial_n^\pm u = \int_\Gamma V\phi(\tfrac{1}{2} \mp K')\phi$$

[Co 2007] Corollary of the "Théorème fondamental"

The operators $A = \frac{1}{2} - K'$ and $B = \frac{1}{2} + K'$ are bounded selfadjoint operators on the space $H^{-\frac{1}{2}}(\Gamma)$ with norm $\|\cdot\|_V$ satisfying $A + B = 1$.

1. A is positive definite, hence B is a contraction, with norm

$$\|B\| \le \frac{\mu}{1+\mu}\,.$$

2. On the subspace $H_0^{-\frac{1}{2}} = \{\phi \mid \langle \phi, 1 \rangle = 0\}$, B is positive definite, hence both A and $N' = A - B$ are contractions, and the Neumann series converges in the norm $\|\cdot\|_V$.

Proof of 1: Poincaré $\Rightarrow \langle V\phi, B\phi \rangle \le \mu \langle V\phi, A\phi \rangle \Rightarrow$
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Same results for $\frac{1}{2} \pm K$ in the space $H^{\frac{1}{2}}(\Gamma)$ with norm defined by the quadratic form of V^{-1}.

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Story 3: Essential spectrum of the double layer potential in 2D

Let $\Lambda = \sigma_{\text{ess}}(K)$ in $H^{\frac{1}{2}}(\Gamma)$.
We have seen (in 3D, but this works in any dimension $n \geq 2$): $\Lambda \subset (\frac{1}{2}, \frac{1}{2})$.

We want to prove for $n = 2$: $\boxed{\Lambda = -\Lambda}$

Remark

It is known that on $\Gamma \subset \mathbb{R}^2$ piecewise smooth with corner angles $\omega_j \in (0, 2\pi)$, Λ is the union of the intervals $[-\frac{|\pi-\omega_j|}{2\pi}, \frac{|\pi-\omega_j|}{2\pi}]$.

Two parts of the story:

1. Motivation:
   Maxwell wave guide problem, formulated in two different ways via volume integral equations;
   Analysis of the VIEs by reduction to boundary integral equations;
   Equivalence between the two formulations requires spectral symmetry
2. Proof of spectral symmetry:
   Boundary integral equation is equivalent to a scalar transmission problem;
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Story 3, Motivation: Maxwell 2D

Dielectric waveguide $\Omega \times \mathbb{R}$, permittivity $\varepsilon = \varepsilon_r \varepsilon_0$, permeability $\mu = \mu_r \mu_0$.
Fields do not depend on x_3.

Equations (TE), valid in $\mathbb{R}^2$, prepared for Volume Integral Equations:

$$\begin{aligned} \partial_1 E_2 - \partial_2 E_1 - i k H_3 &= ik(\mu_r - 1)\chi_\Omega H_3 \,; \\ \partial_2 H_3 + i k E_1 &= ik(1 - \varepsilon_r)\chi_\Omega E_1 + J_1 \,; \\ -\partial_1 H_3 + i k E_2 &= ik(1 - \varepsilon_r)\chi_\Omega E_2 + J_2 \,. \end{aligned}$$

1. Electric formulation (elimination of H_3):

$$\mathbf{curl}\, \mathrm{curl}\, \boldsymbol{E} - k^2 \boldsymbol{E} = \mathbf{curl}\left(\tfrac{\mu_r - 1}{\mu_r}\chi_\Omega\, \mathrm{curl}\, \boldsymbol{E}\right) + ik(1 - \varepsilon_r)\chi_\Omega \boldsymbol{E} + \boldsymbol{J} \,.$$

2. Magnetic formulation (elimination of $\boldsymbol{E} = (E_1, E_2)$, assumption $\chi_\Omega \boldsymbol{J} = 0$):

$$-(\Delta + k^2) H_3 = \mathrm{curl}\left(\tfrac{\varepsilon_r - 1}{\varepsilon_r}\chi_\Omega\, \mathbf{curl}\, H_3\right) - ik(\mu_r - 1)\chi_\Omega H_3 \,.$$

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Equations (TE), valid in $\mathbb{R}^2$, prepared for Volume Integral Equations:

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\partial_1 E_2 - \partial_2 E_1 - i\,kH_3 &= ik(\mu_r - 1)\chi_\Omega H_3 \;; \\
\partial_2 H_3 + i\,kE_1 &= ik(1-\varepsilon_r)\chi_\Omega E_1 + J_1 \;; \\
-\partial_1 H_3 + i\,kE_2 &= ik(1-\varepsilon_r)\chi_\Omega E_2 + J_2 \;.
\end{aligned}$$

1. Electric formulation (elimination of H_3):

$$\mathbf{curl}\,\mathrm{curl}\,\boldsymbol{E} - k^2\boldsymbol{E} = \mathbf{curl}\left(\tfrac{\mu_r - 1}{\mu_r}\chi_\Omega\,\mathrm{curl}\,\boldsymbol{E}\right) + ik(1-\varepsilon_r)\chi_\Omega\boldsymbol{E} + \boldsymbol{J}\,.$$

2. Magnetic formulation (elimination of $\boldsymbol{E} = (E_1, E_2)$, assumption $\chi_\Omega \boldsymbol{J} = 0$):

$$-(\Delta + k^2)H_3 = \mathrm{curl}\left(\tfrac{\varepsilon_r - 1}{\varepsilon_r}\chi_\Omega\,\mathbf{curl}\,H_3\right) - ik(\mu_r - 1)\chi_\Omega H_3\,.$$

For simplicity: $\mu_r \equiv 1$, $\boldsymbol{J} \equiv 0$.

(PDE) $$\mathbf{curl}\,\mathrm{curl}\,\boldsymbol{E} - k^2\boldsymbol{E} = ik(1-\varepsilon_r)\chi_\Omega \boldsymbol{E}$$

Convolution with fundamental solution (+ radiation condition)

$$k^{-2}(\nabla\,\mathrm{div} + k^2)g_k\,;\quad g_k(x) = \frac{i}{4}H_0^{(1)}(k|x|)$$

(VIE) $$\boldsymbol{E} + (\nabla\,\mathrm{div} + k^2)g_k * (1-\varepsilon_r)\chi_\Omega \boldsymbol{E} = \boldsymbol{E}^0$$

Analyze the strongly singular volume integral equation $(1 + A_k)\boldsymbol{E} = \boldsymbol{E}^0$ with

$$A_k\boldsymbol{E} = (\nabla g_k * \mathrm{div} + k^2 g_k *)(1-\varepsilon_r)\chi_\Omega \boldsymbol{E}$$

Simplification: $k = 0$, $\varepsilon_r = \mathrm{const}$, $\mathrm{div}\,\boldsymbol{E} = 0$ (On $\nabla H_0^1(\Omega)$, $A_0 = (\varepsilon_r - 1)\mathbb{I}$):

$$A_0\boldsymbol{E} = (\varepsilon_r - 1)\nabla\mathscr{S}(\boldsymbol{n}\cdot\boldsymbol{E})$$

Spectrally equivalent to (interchanging $\nabla\mathscr{S}$ and $\boldsymbol{n}\cdot$)

$$(\varepsilon_r - 1)\partial_n\mathscr{S} = (\varepsilon_r - 1)(\tfrac{1}{2} + K')$$

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[Costabel-Darrigrand-Sakly 2015]

The "electric" VIE is Fredholm in $\boldsymbol{H}(\mathbf{curl},\Omega)$ if and only if $\varepsilon_r \neq 0$ and the boundary integral operator

$$1-(1-\varepsilon_r)(\tfrac{1}{2}+K')$$

is Fredholm in $H^{-\frac{1}{2}}(\Gamma)$.

In our notation, the condition is

Fredholmness condition, electric VIE

$$\frac{1+\varepsilon_r}{2(1-\varepsilon_r)} \notin \Lambda \qquad \Longleftrightarrow \qquad \varepsilon_r \neq \frac{2\lambda-1}{2\lambda+1} \quad \forall \lambda \in \Lambda$$

(PDE) $$-(\Delta + k^2)H_3 = \operatorname{curl}\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\chi_\Omega \,\mathbf{curl}\, H_3\right)$$

Story 3, Magnetic VIE

(PDE) $$-(\Delta + k^2)H_3 = \operatorname{curl}\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\chi_\Omega\, \mathbf{curl}\, H_3\right)$$

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(VIE) $$H_3 - \operatorname{curl} g_k * \left(\frac{\varepsilon_r - 1}{\varepsilon_r}\chi_\Omega\, \mathbf{curl}\, H_3\right) = H_3^0$$

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Task

Analysis of the strongly singular volume integral operator
(for $k = 0$ and $\varepsilon_r = \text{const}$)

$$\operatorname{curl} g_k * (\chi_\Omega \,\mathbf{curl}) = 1 + \mathscr{D}\gamma$$

Thus $1 + \frac{\varepsilon_r - 1}{\varepsilon_r}\operatorname{curl} g_k * (\chi_\Omega \,\mathbf{curl})$ on $H^1(\Omega)$ is equivalent to

$$1 - \frac{\varepsilon_r - 1}{\varepsilon_r}\left(\tfrac{1}{2} + K\right) = \frac{\varepsilon_r - 1}{\varepsilon_r}\left(\frac{\varepsilon_r + 1}{2(\varepsilon_r - 1)} - K\right) \qquad \text{on } H^{\frac{1}{2}}(\Gamma)$$

Fredholmness condition: magnetic VIE

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Story 3, Proof of $\Lambda = -\Lambda$, without integral operators

We consider a larger domain

$$\hat{\Omega} = \Omega \cup \Gamma \cup \hat{\Omega}^+$$

The transmission problem (with Dirichlet condition on $\partial\hat{\Omega}$)

$$\operatorname{div} \varepsilon \nabla u = f\,, \quad u \in H_0^1(\hat{\Omega})\,, \quad \varepsilon = \begin{cases} \varepsilon_r & \text{in } \Omega \\ 1 & \text{in } \hat{\Omega}^+ \end{cases}$$

can be reduced to a boundary integral equation using a single layer potential via

$$u = w + \mathscr{S}\psi \quad \text{with } w \in H_0^1(\Omega) \oplus H_0^1(\hat{\Omega}^+)\,, \quad \varepsilon \Delta w = f$$

The integral operator corresponds to the jump in the conormal derivative

$$(\varepsilon_r \partial_n^- - \partial_n^+)\mathscr{S} = \varepsilon_r(\tfrac{1}{2} + K') - (-\tfrac{1}{2} + K') = \tfrac{1}{2}(\varepsilon_r + 1) - (1 - \varepsilon_r)K'$$

Proposition

$\operatorname{div} \varepsilon \nabla : H_0^1(\hat{\Omega}) \to H^{-1}(\hat{\Omega})$ is Fredholm if and only if

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$$\frac{1 + \varepsilon_r}{2(1 - \varepsilon_r)} \notin \Lambda\,.$$

The proof is concluded using the

Lemma [Bonnet-Ben Dhia – Chesnel – Ciarlet 2014, Co–Dar–Sak 2015]

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let the complex-valued coefficient function $\varepsilon \in L^\infty(\Omega)$ satisfy $\frac{1}{\varepsilon} \in L^\infty(\Omega)$. Then

$$\operatorname{div} \varepsilon \nabla : \; H_0^1(\Omega) \to H^{-1}(\Omega)$$

is Fredholm if and only if

$$\operatorname{div} \tfrac{1}{\varepsilon} \nabla : \; H^1(\Omega) \to \big(H^1(\Omega)\big)'$$

is Fredholm.

Thus $\quad \frac{1+\varepsilon_r}{2(1-\varepsilon_r)} \notin \Lambda \iff \frac{1+\frac{1}{\varepsilon_r}}{2(1-\frac{1}{\varepsilon_r})} \notin \Lambda$

But $\quad \lambda = \frac{1+\varepsilon_r}{2(1-\varepsilon_r)} \Longrightarrow \frac{1+\frac{1}{\varepsilon_r}}{2(1-\frac{1}{\varepsilon_r})} = -\lambda \qquad$ CQFD.

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Thank you for your attention!

All the best, Patrick, for the second 60 years !

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