Computing the inf-sup constant of the divergence

Martin Costabel

Collaboration with Monique Dauge
with contributions from C. Bernardi, V. Girault, M. Crouzeix, Y. Lafranche

IRMAR, Université de Rennes 1

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Publications

- M. Dauge, C. Bernardi, M. Costabel, V. Girault
  On Friedrichs constant and Horgan-Payne angle for LBB condition
  Monogr. Mat. Garcia de Galdeano (2014)
- M. Costabel, M. Dauge
  On the inequalities of Babuška–Aziz, Friedrichs and Horgan–Payne
  (2013) http://fr.arxiv.org/abs/1303.6141
- M. Costabel, M. Crouzeix, M. Dauge, Y. Lafranche
  The inf-sup constant for the divergence on corner domains
  Num. Meth. for Partial Diff. Eq. (2014).

- Ω bounded domain in $\mathbb{R}^d$ ($d \geq 1$). No regularity assumptions.

The inf-sup constant of Ω

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_1 \, \|q\|_0}$$

- $L^2(\Omega)$ space of square integrable functions q on Ω. Norm $\|q\|_0$
- $H^1(\Omega)$ Sobolev space of $v \in L^2(\Omega)$ with gradient $\nabla v \in L^2(\Omega)^d$
- $L^2_\circ(\Omega)$ subspace of $q \in L^2(\Omega)$ with $\int_\Omega q = 0$.
- $H^1_0(\Omega)$ closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$ (zero trace on $\partial\Omega$) (Semi-)Norm $|v|_1 = \|\nabla v\|_0$ equivalent to norm $\|v\|_{H^1(\Omega)}$
- $\beta(\Omega)$ is invariant with respect to translations, rotations, dilations.
- We will often talk about $\boxed{\sigma(\Omega) = \beta(\Omega)^2}$ instead of $\beta(\Omega)$.

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The inf-sup Constant: Known Values

Ball in $\mathbb{R}^d$: $\sigma(\Omega) = \dfrac{1}{d}$ [Ellipsoids in 3D: E.&F. Cosserat 1898]

In 2D:

Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} < 1$, $a < b$: $\sigma(\Omega) = \dfrac{a^2}{a^2 + b^2}$

Some other domains with simple conformal mappings, for example:

Annulus $a < r < 1$: $\sigma(\Omega) = \dfrac{1}{2} - \dfrac{1}{2}\sqrt{\dfrac{1-a^2}{1+a^2}\,\dfrac{1}{\log 1/a}}$

[Chizhonkov-Olshanskii 2000]

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Always true: $0 \le \beta(\Omega) \le 1$

Any bounded Lipschitz domain Ω: $\beta(\Omega) > 0$
["inf-sup condition", "LBB condition"]

Any finite union of bounded star-shaped domains: $\beta(\Omega) > 0$

Inward cusps, cracks: OK, $\beta(\Omega) > 0$

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Any bounded **John domain** Ω: $\beta(\Omega) > 0$ [R. Durán et al. 2006]

Rectangle of aspect ratio $a \ll 1$: $\dfrac{a^2}{4} \le \sigma(\Omega_a) \le \dfrac{\pi^2}{12} a^2$ and

$$\sigma(\Omega_a) \le \frac{1}{2} - \frac{1}{\pi} \approx 0.18169$$

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The inf-sup Constant: The square

The square $\Omega = (0,1) \times (0,1) =: \square \subset \mathbb{R}^2$

The Square

$\beta(\square)$ is currently still unknown !

Conjecture 1

$$\sigma(\square) = \frac{2}{7} \approx 0.2857\ldots \qquad (\to\ \beta(\square) \approx 0.5345)$$

Conjecture 2

$$\sigma(\square) = \frac{1}{2} - \frac{1}{\pi} \approx 0.18169\ldots \qquad (\to\ \beta(\square) \approx 0.42625)$$

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C. O. Horgan and L. E. Payne, *On inequalities of Korn, Friedrichs and Babuška-Aziz*. Arch. Rational Mech. Anal., **82** (1983), pp. 165–179.

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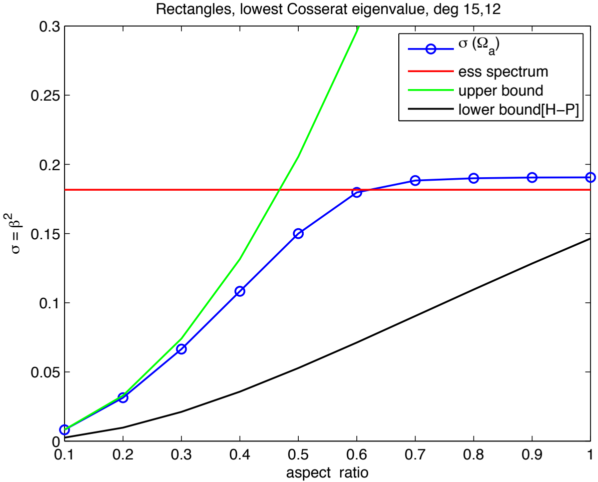
Computation on rectangles with aspect ratio 0.1 . . . 1
80 elements (Q_{15}, Q_{12}), $\sim$ 30000 dof
First Cosserat eigenvalue (computed with a Stokes solver)

- $\sigma(\Omega) = \beta(\Omega)^2$ is the minimum of the Cosserat spectrum



Consider the Stokes problem for $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$, $p \in L^2_\circ(\Omega)$:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega \end{aligned}$$

Pressure Stability for the Stokes problem

Let Ω be such that $\beta(\Omega) > 0$. Let C_P be the constant in the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \le C_P \|\nabla v\|_{L^2(\Omega)} \qquad \forall\, v \in H^1_0(\Omega).$$

Then for $f \in \boldsymbol{L}^2(\Omega)$ there exists a unique solution $(\boldsymbol{u}, p)$ of the Stokes problem, and

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \le C_P \|f\|_{L^2(\Omega)}$$

$$\|p\|_{L^2(\Omega)} \le \frac{2C_p}{\beta(\Omega)} \|f\|_{L^2(\Omega)}$$

Computing $\beta(\Omega)$: The Discrete inf-sup Constant

- Conforming discretization: $V_N \subset H_0^1(\Omega)^d$, $M_N \subset L^2_\circ(\Omega)$

The discrete inf-sup constant of (V_N, M_N)

$$\beta_N = \inf_{q \in M_N} \sup_{\boldsymbol{v} \in V_N} \frac{\int_\Omega \operatorname{div} \boldsymbol{v} \, q}{|\boldsymbol{v}|_1 \, \|q\|_0}$$

Galerkin approximation of the Stokes system

$(\boldsymbol{u}_N, p_N) \in V_N \times M_N$:

$$\int_\Omega \nabla \boldsymbol{u}_N \cdot \nabla \boldsymbol{v} - \int_\Omega \operatorname{div} \boldsymbol{v} \, p_N = \int_\Omega \mathbf{f} \cdot \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in V_N$$

$$\int_\Omega \operatorname{div} \boldsymbol{u}_N \, q = 0 \quad \forall\, q \in M_N$$

Necessary and sufficient condition for stability of Galerkin scheme

(Babuška-Brezzi) $\quad \inf_N \beta_N = \beta_0 > 0$

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Continuous vs Discrete inf-sup Constant

In general, one can have $\boxed{\beta_N \leq \beta(\Omega)}$ or $\boxed{\beta_N \geq \beta(\Omega)}$.

No general rule known.

Lemma

If $(M_N)_N$ approximates $L^2_\circ(\Omega)$, then

$$\boxed{\limsup_N \beta_N \leq \beta(\Omega)}$$

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Proof: Let $q \in L^2_\circ(\Omega)$, $q_N \in M_N$, $q_N \to q$, $\beta_N \to \beta_\infty$.

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$$\Longrightarrow \frac{\langle \operatorname{div} \boldsymbol{v}_N, q\rangle_\Omega}{|\boldsymbol{v}_N|_1} = \frac{\langle \operatorname{div} \boldsymbol{v}_N, q_N\rangle_\Omega}{|\boldsymbol{v}_N|_1} - \frac{\langle \operatorname{div} \boldsymbol{v}_N, q_N - q\rangle_\Omega}{|\boldsymbol{v}_N|_1}$$

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Well-studied question: Bound β_N from below

For *p* and *hp* version:

Triangles: Vogelius 1983, Scott-Vogelius 1985

Quads: Bernardi-Maday 1999
Stenberg-Suri 1996
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Ainsworth-Coggins 2000, 2002

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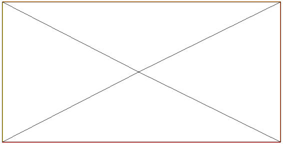
Our question here: When is $\boxed{\lim_N \beta_N = \beta(\Omega)}$?

Rectangle of aspect ratio $a = 0.5$: $\sigma(\Omega) = 0.14996..$



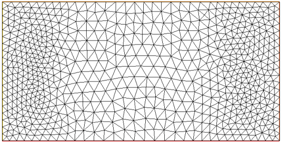
Almost singular "Four Corners" triangulation

Scott-Vogelius $P_4 - P_3^{dc}$: $\sigma_N = 0.000021$

$P_4 - P_3$: $\sigma_N = 0.149769$

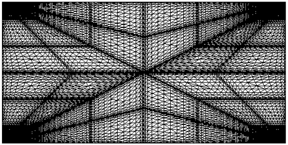
$P_4 - P_2^{dc}$: $\sigma_N = 0.153970$

$P_4 - P_2$: $\sigma_N = 0.154518$



Refined mesh, $P_4 - P_2$: $\sigma_N = 0.151573$

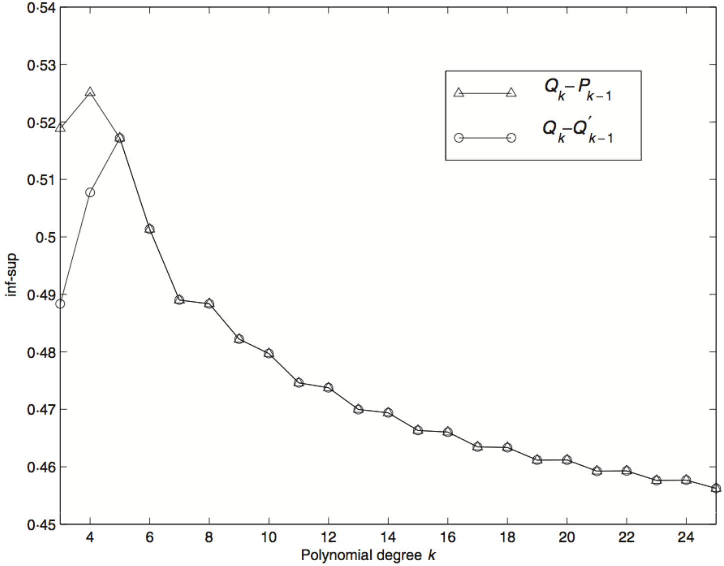
(Freefem++)



Geometric refinement at corners,

$Q_{16} - Q_{14}$: $\sigma_N = 0.149960$

(Melina++)



Variation of inf–sup constants for $\mathcal{Q}_k - \mathcal{P}_{k-1}$ and $\mathcal{Q}_k - \mathcal{Q}^{'}_{k-1}$ methods on the reference element $\hat{K}$.

Square according to [Ainsworth-Coggins 2002] $\beta(\Omega) \leq 0.42625$.

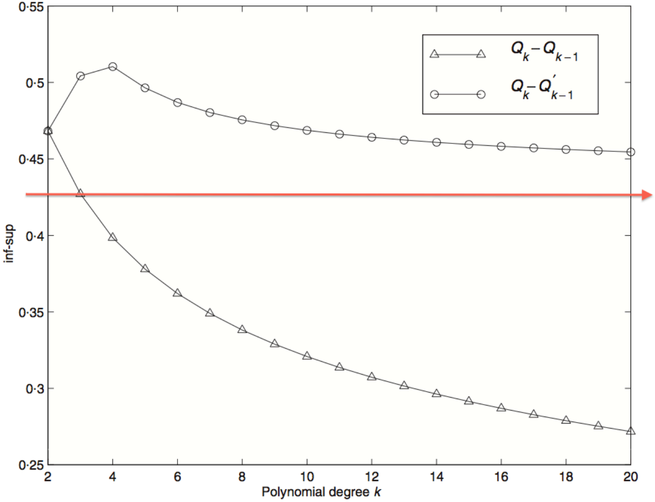
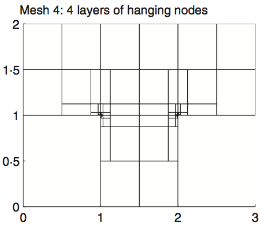
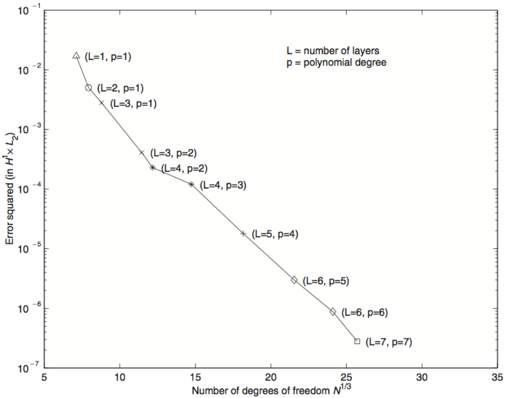


FIG. 3. Inf-sup constants for the generalized Taylor–Hood elements $\mathcal{Q}_k - \mathcal{Q}_{k-1}$ and the new family $\mathcal{Q}_k - \mathcal{Q}'_{k-1}$ analysed in the text.

hp for Stokes according to [Ainsworth-Coggins 2002]





Relation with the Cosserat eigenvalue problem

[Eugène & François Cosserat 1898]

Find $\boldsymbol{u} \in H_0^1(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$ such that

$$\sigma \Delta \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u} = 0\,.$$

Their aim: Solving the Lamé Dirichlet problem by eigenfunction expansion.

For $\sigma \neq 0$, equivalent eigenvalue problem:

$$\operatorname{div} \Delta^{-1} \nabla q = \sigma q \qquad \text{in } L^2_\circ(\Omega).$$

Definition: Cosserat operator $\mathcal{S} = \operatorname{div} \Delta^{-1} \nabla$ Selfadjoint, positive, ≤ 1.

The Cosserat eigenvalue problem is a Stokes eigenvalue problem

Find $\boldsymbol{u} \in H_0^1(\Omega)$, $p \in L^2_\circ(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

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Definition: Cosserat operator $\boxed{\mathscr{S} = \operatorname{div} \Delta^{-1} \nabla}$ Selfadjoint, positive, ≤ 1.

The Cosserat eigenvalue problem is a Stokes eigenvalue problem

Find $\boldsymbol{u} \in H_0^1(\Omega)$, $p \in L^2_\circ(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

$$\boxed{\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= 0 \qquad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= \sigma p \qquad \text{in } \Omega \end{aligned}}$$

This is not an elliptic eigenvalue problem! $\sigma = 1$ has infinite multiplicity

$$q = \Delta\phi,\ \phi \in H_0^2(\Omega) \Rightarrow \Delta^{-1}\nabla q = \nabla\phi \Rightarrow \mathscr{S} q = q.$$

Define $\boxed{\sigma(\Omega) = \min(\text{ Spectrum } \mathscr{S})}$

Known results [Cosserats, Nečas, Maz'ya–Mikhlin]

Ball in $\mathbb{R}^d$: $\sigma(\Omega) = \dfrac{1}{d}$, $\sigma_k = \frac{k}{2k+d-2}$, $k \geq 1$

Bounded Lipschitz domains: $\sigma(\Omega) > 0$

- $\sigma = 1$ is an isolated eigenvalue,
- $\sigma = \frac{1}{2}$ is accumulation point of eigenvalues

Smooth domains (C^3 [Crouzeix 1997]):

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A simple relation

$$\sigma(\Omega) = \beta(\Omega)^2$$

Relation with the Cosserat eigenvalue problem

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Proof

Advantages

- Standard code available: Stokes + matrix eigenvalue problem
- Eigenfunctions can be looked at

Disadvantage

There is no theory for the approximation of this eigenvalue problem.

Stokes eigenvalue problem, first kind

Find $\boldsymbol{u} \in H_0^1(\Omega)$, $p \in L_\circ^2(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \sigma\, \boldsymbol{u} \quad &\text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= 0 \quad &\text{in } \Omega \end{aligned}$$

Known: Discrete LBB condition guarantees spectral convergence.

Stokes eigenvalue problem, second kind

Find $\boldsymbol{u} \in H_0^1(\Omega)$, $p \in L_\circ^2(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= 0 \quad &\text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= \sigma\, p \quad &\text{in } \Omega \end{aligned}$$

No convergence proof known.

Reformulation as Cosserat or Stokes eigenvalue problem

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- Standard code available: Stokes + matrix eigenvalue problem
- Eigenfunctions can be looked at

Disadvantage

There is no theory for the approximation of this eigenvalue problem.

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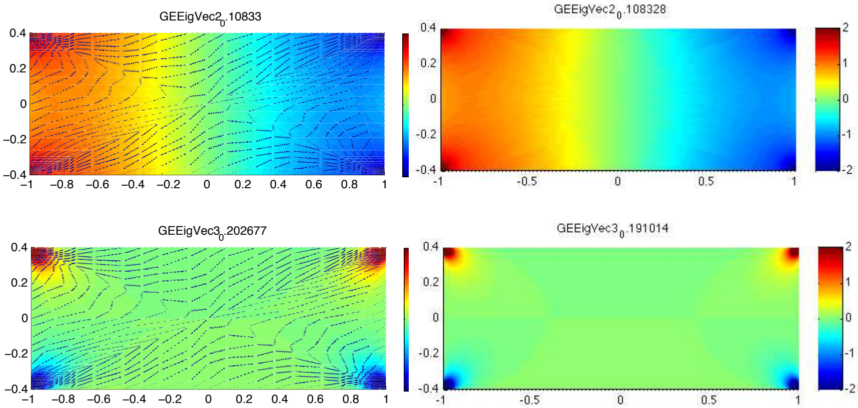
No convergence proof known.

Rectangle, aspect ratio 0.4

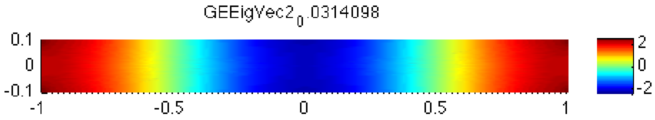
Degrees: 6, 3

Degrees: 15,12



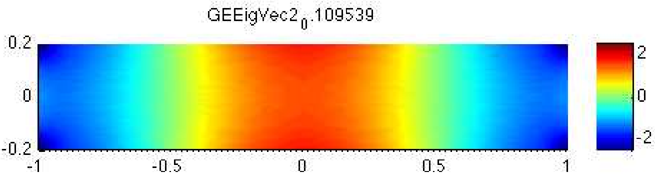
Rectangle: First eigenfunction

Rectangle, aspect ratio 0.1

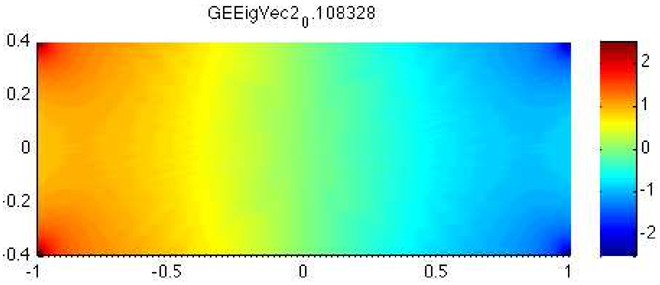


Rectangle: First eigenfunction

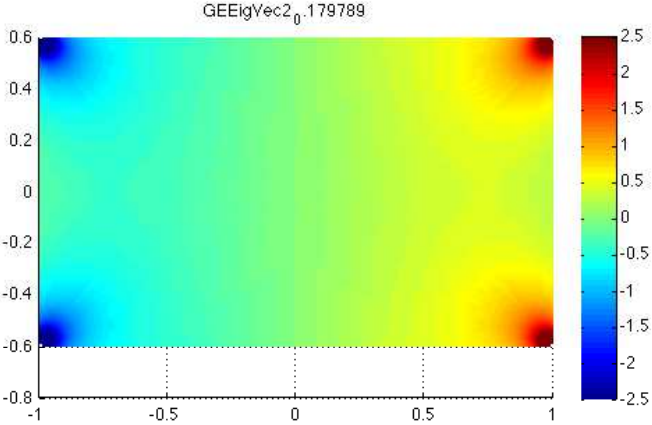
Rectangle, aspect ratio 0.2



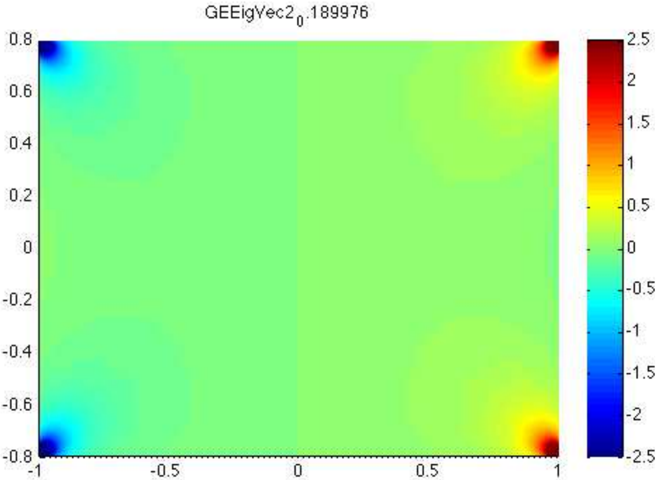
Rectangle, aspect ratio 0.4



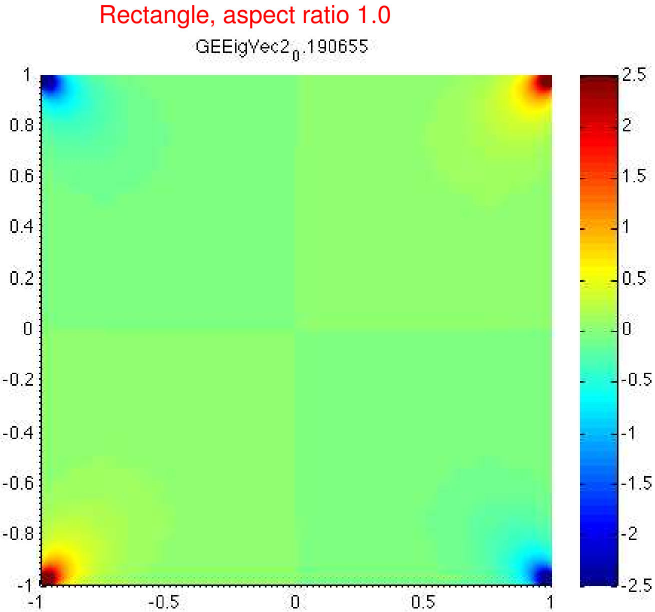
Rectangle, aspect ratio 0.6



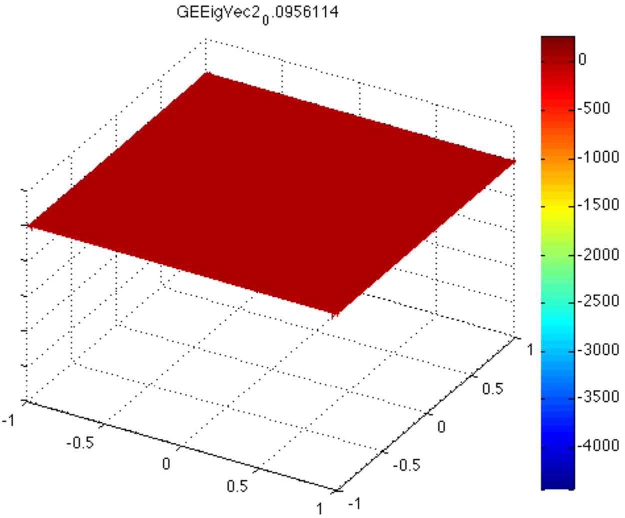
Rectangle, aspect ratio 0.8



Rectangle: First eigenfunction



Square: First eigenfunction, (Q_{17}, Q_{16})



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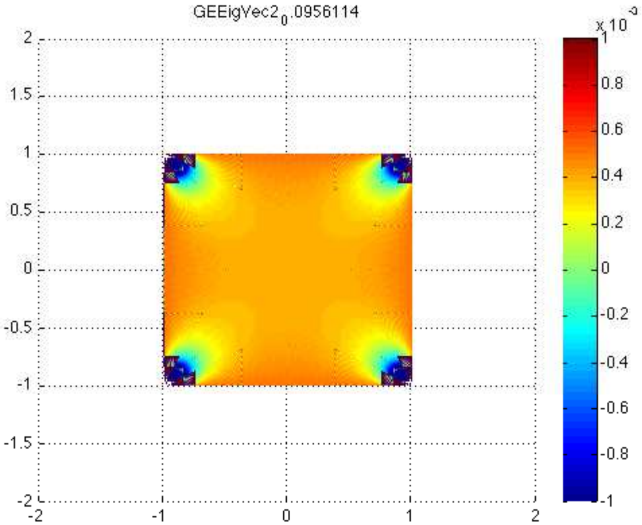

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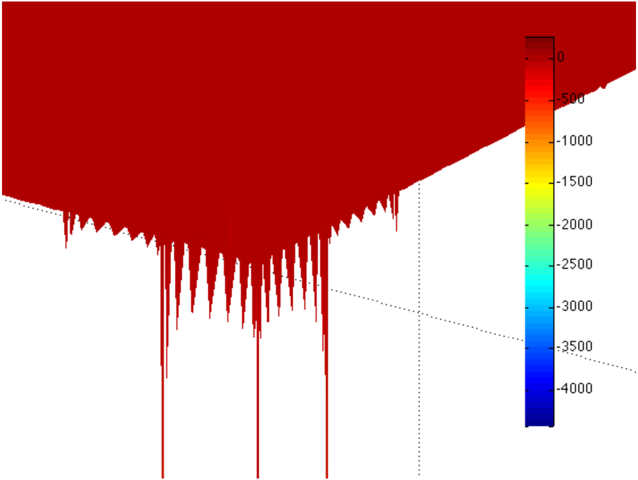
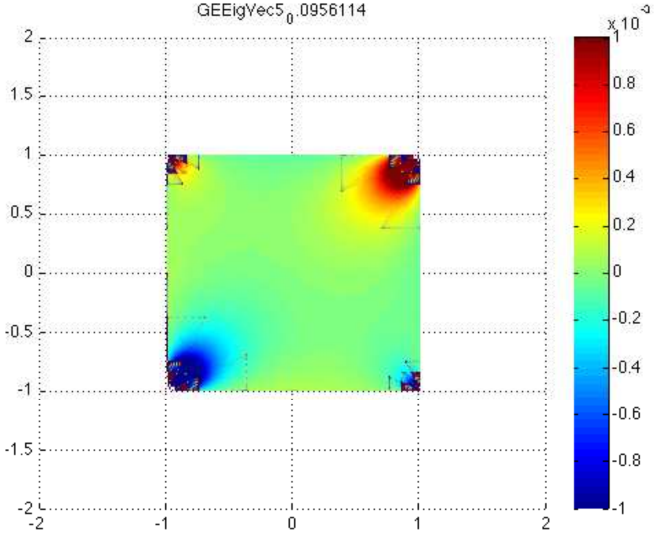


Square: Fourth eigenfunction, (Q_{17}, Q_{16})



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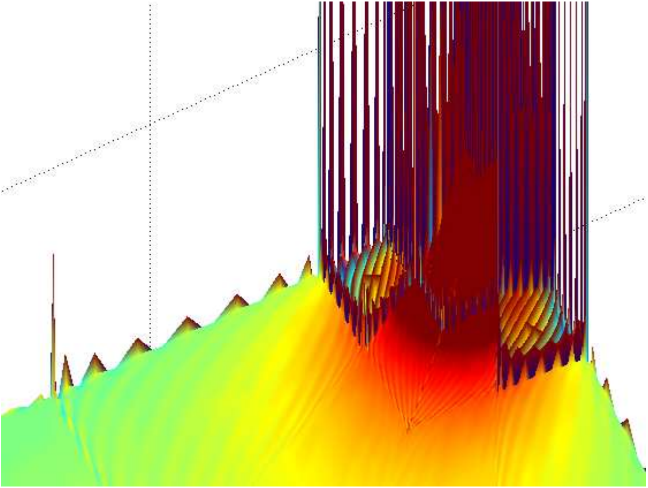
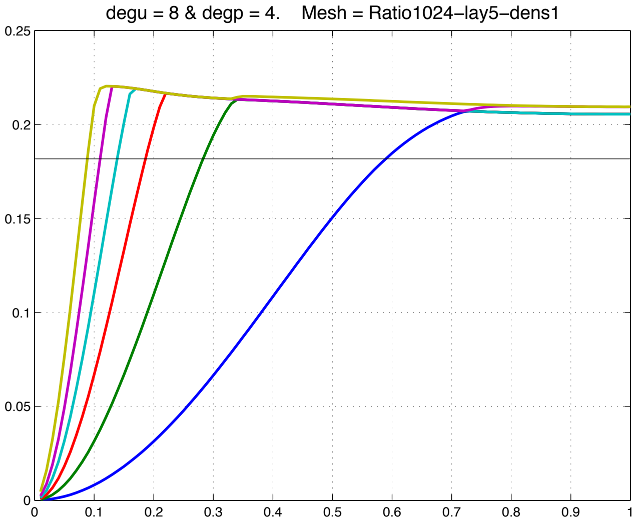
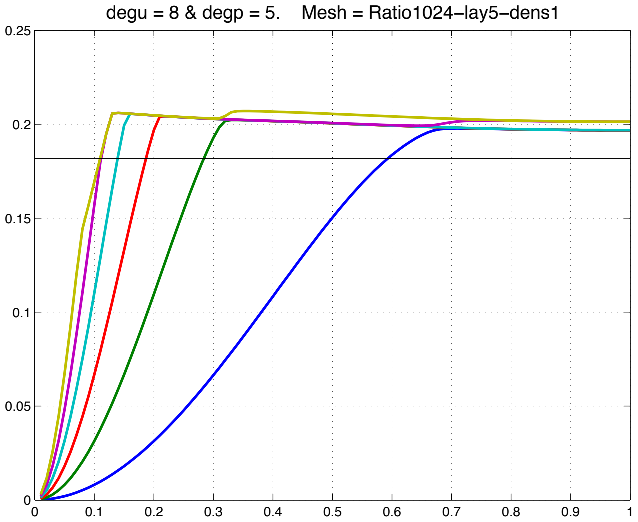


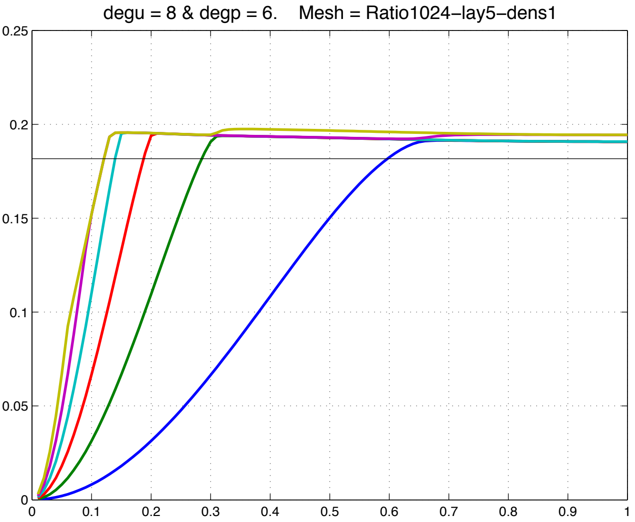
$k = 8$ and $\ell = 4$. Strongly refined mesh



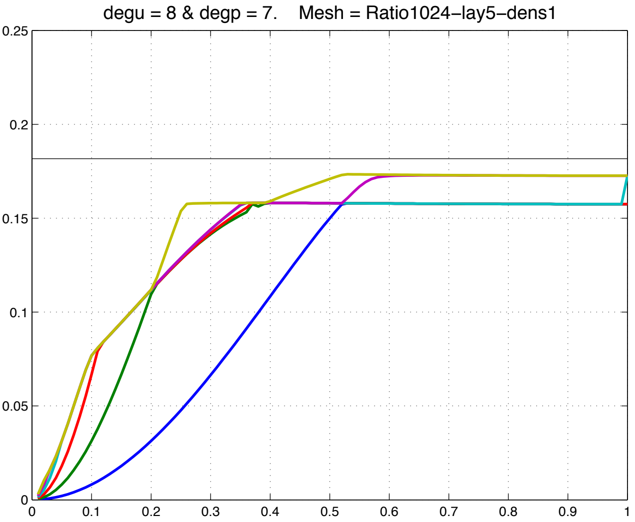
$k = 8$ and $\ell = 5$. Strongly refined mesh



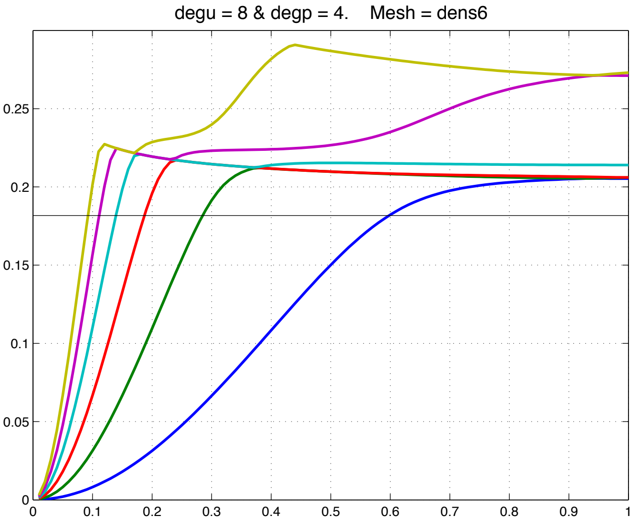
$k = 8$ and $\ell = 6$. Strongly refined mesh



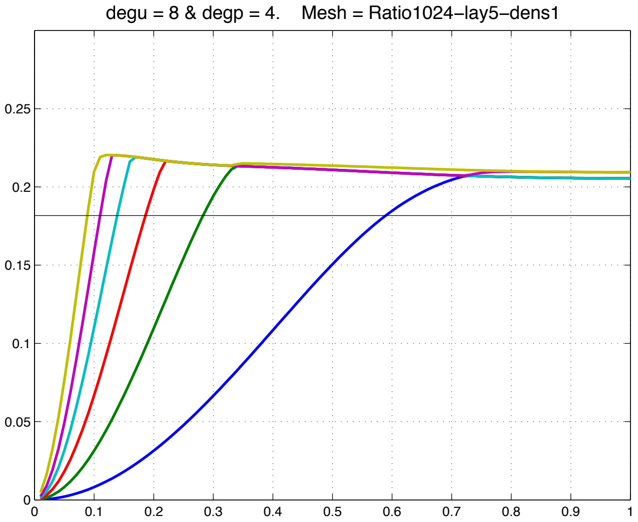
$k = 8$ and $\ell = 7$. Strongly refined mesh



Again, for comparison: $k = 8$ and $\ell = 4$. Uniform grid



Again, for comparison: $k = 8$ and $\ell = 4$. Refin. mesh



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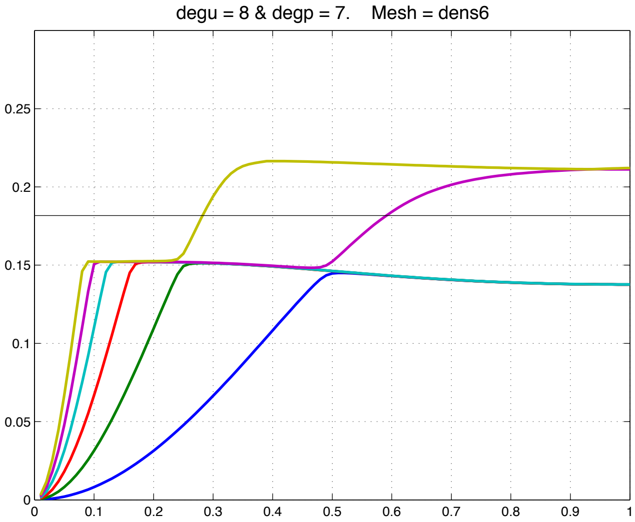

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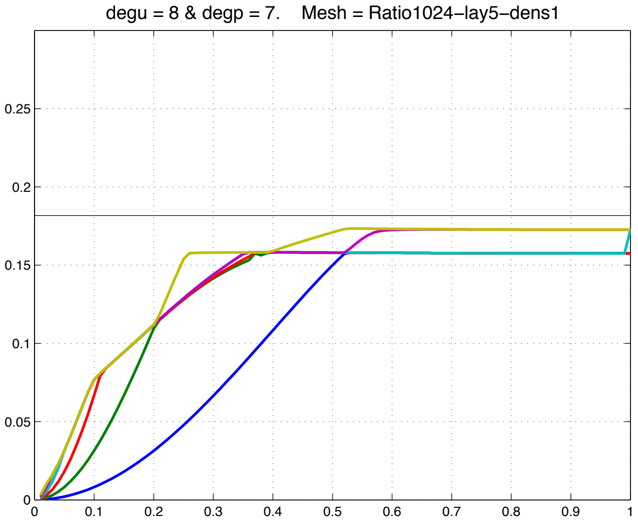

Corner singularities

For $\sigma \notin \{0, \frac{1}{2}, 1\}$, the operator $A_\sigma = -\sigma\Delta + \nabla \operatorname{div}$ is elliptic.
If $\Omega \subset \mathbb{R}^2$ has a corner of opening ω, one can therefore determine the corner singularities via **Kondrat'ev**'s method of Mellin transformation:
Look for solutions of the form $r^\lambda \phi(\theta)$ in a sector. $\rightarrow q \sim r^{\lambda-1}\phi(\theta)$
Characteristic equation (Lamé system, known!) for a corner of opening ω:

$$(*) \qquad (1-2\sigma)\frac{\sin \lambda\omega}{\lambda} = \pm \sin\omega.$$

Theorem [Kondrat'ev 1967]

For $\sigma \in [0,1] \setminus \{0, \frac{1}{2}, 1\}$, $A_\sigma : H_0^1(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$ is Fredholm iff the equation $(*)$ has no solution on the line $\Re e\, \lambda = 0$.

With $z = \lambda\omega$, we rewrite $(*)$:

$$(1-2\sigma)\frac{\sin z}{z} = \pm\frac{\sin\omega}{\omega}.$$

Result :

- $(*)$ has roots on the line $\Re e\, \lambda = 0$ iff $|1-2\sigma|\omega < |\sin\omega|$
- If $|1-2\sigma|\omega > |\sin\omega|$, there is a real root $\lambda \in (0,1)$

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Essential spectrum: Corners

Theorem [Crouzeix, Costabel-Dauge]

$\Omega \subset \mathbb{R}^2$ piecewise smooth with corners of opening ω_j.

$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}) = \bigcup_{\text{corners } j} \left[\tfrac{1}{2} - \frac{|\sin\omega_j|}{2\omega_j}, \tfrac{1}{2} + \frac{|\sin\omega_j|}{2\omega_j}\right] \cup \{1\}$$

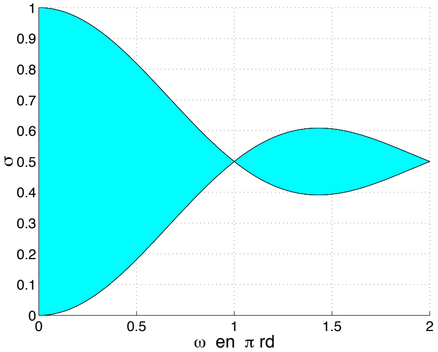


Figure: Essential spectrum: σ vs. opening ω

Example : Rectangle, $\omega = \frac{\pi}{2}$

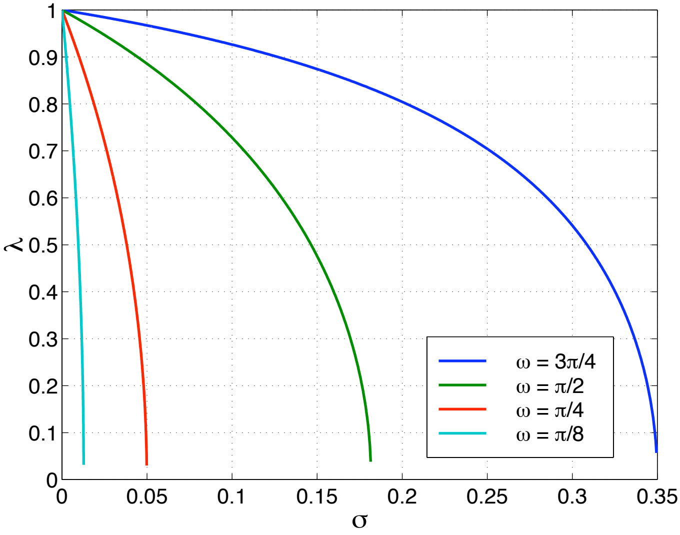
$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}\Big|_M) = [\tfrac{1}{2} - \tfrac{1}{\pi}, \tfrac{1}{2} + \tfrac{1}{\pi}]$$
$$\approx [0.181, 0.818]$$

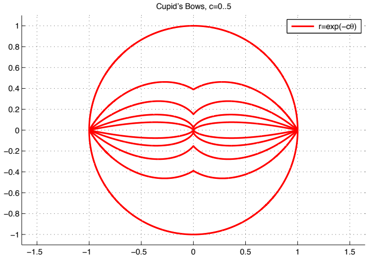
Corollary

For square, rectangles, rectangular cylinders in 3D:

$$\beta(\Omega)^2 \le \tfrac{1}{2} - \tfrac{1}{\pi}$$

Exponent of singularity vs Cosserat eigenvalue (Rectangle: green line)





Logarithmic spirals: $r = e^{-c\theta}, 0 \le \theta \le \frac{\pi}{2}$ + symmetries

Horgan-Payne angle: Minimal angle between radius vector and tangent

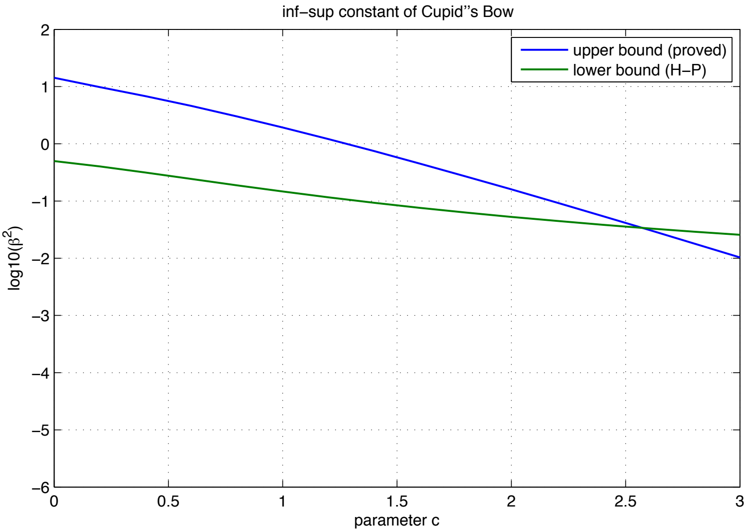
$$\omega(\Omega) = \arctan \frac{1}{c}$$

Horgan-Payne inequality: $\boxed{\beta(\Omega) \ge \sin \frac{\omega(\Omega)}{2}}$

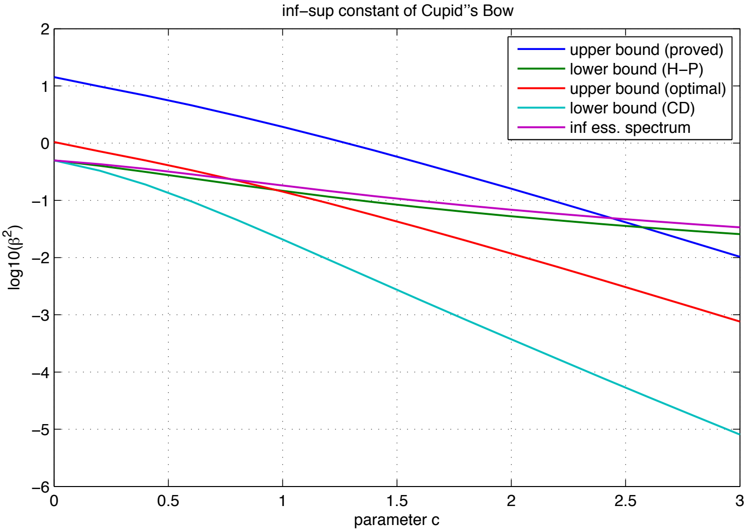
$$\beta(\Omega)^2 \ge \frac{\sqrt{c^2+1}-c}{2\sqrt{c^2+1}} = \frac{1}{4c^2} + O(c^{-4}) \quad \text{as } c \to \infty .$$

Upper bound [Costabel-Dauge 2013]

$$\beta(\Omega)^2 \le \frac{128}{3} \frac{c\, e^{-c\pi}}{1-e^{-c\pi}} \qquad \left(\tfrac{128}{3} \to \pi \text{ [Co-Da-Crouzeix]} \right)$$


inf−sup constant of Cupid''s Bow
upper bound (proved)
lower bound (H−P)
log10(β²)
parameter c
2
1
0
−1
−2
−3
−4
−5
−6
0
0.5
1
1.5
2
2.5
3

Cupid’s Bow


inf−sup constant of Cupid’’s Bow
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parameter c
2
1
0
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−3
−4
−5
−6
0
0.5
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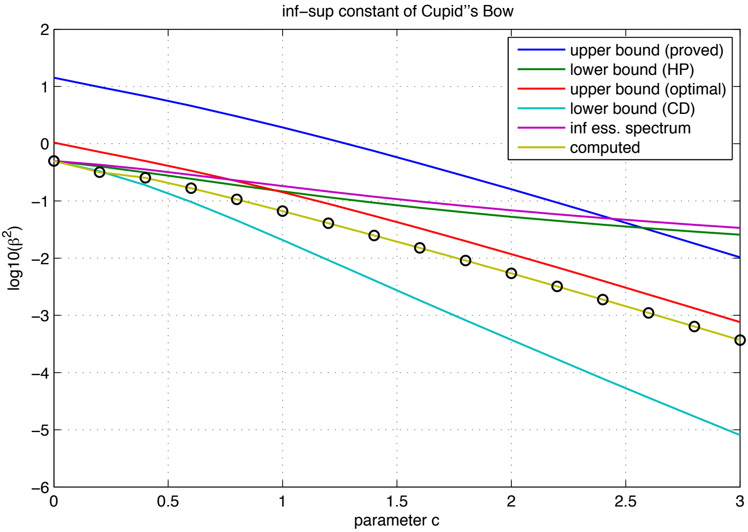
2
2.5
3
upper bound (proved)
lower bound (H−P)
upper bound (optimal)
lower bound (CD)
inf ess. spectrum

Cupid’s Bow


inf−sup constant of Cupid''s Bow
upper bound (proved)
lower bound (HP)
upper bound (optimal)
lower bound (CD)
inf ess. spectrum
computed
log10(β²)
parameter c
2
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−2
−3
−4
−5
−6
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0.5
1
1.5
2
2.5
3

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Lions' lemma

Lemma [Lions 1958, unpublished*, for smooth domains] [Nečas 1967 for Lipschitz domains]

$$\|q\|_0^2 \leq C(\Omega)\, |\nabla q|_{-1}^2 \qquad \forall q \in L^2_\circ(\Omega)$$

* According to [E. Magenes and G. Stampacchia 1958].

$$\rightarrow \quad C(\Omega) = \frac{1}{\beta(\Omega)^2}$$

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$H^{-1}(\Omega)$ dual space of $H^1_0(\Omega)$ with dual norm $|\cdot|_{-1}$:

For $q \in L^2_\circ(\Omega)$:

$$|\nabla q|_{-1} = \sup_{\boldsymbol{v}\in H^1_0(\Omega)^d} \frac{\langle \nabla q, \boldsymbol{v}\rangle_\Omega}{|\boldsymbol{v}|_1} = \sup_{\boldsymbol{v}\in H^1_0(\Omega)^d} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_{1,\Omega}}$$

$$\beta(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \frac{|\nabla q|_{-1}}{\|q\|_0}$$

$$\to \boxed{C(\Omega) = \frac{1}{\beta(\Omega)^2}}$$

Lions' Lemma and Right Inverse of the Divergence

Lions' Lemma $\iff \nabla \quad : L^2_\circ(\Omega) \to H^{-1}(\Omega)^d$ is injective with closed range
$\iff \operatorname{div} : H^1_0(\Omega)^d \to L^2_\circ(\Omega)$ is surjective

Babuška-Aziz inequality [Babuška-Aziz 1971], named by [Horgan-Payne 1983]

Ω Lipschitz, $q \in L^2_\circ(\Omega) \quad \Longrightarrow \quad \exists\, \boldsymbol{v} \in H^1_0(\Omega)^d : \operatorname{div} \boldsymbol{v} = q$

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Equivalence for any domain Ω:

$$\beta(\Omega) > 0 \iff \text{Lions' lemma} \iff \text{Babuška-Aziz inequality}$$

This condition (and its discrete counterpart) is called inf-sup condition or LBB condition, after

- **L**adyzhenskaya — Added by J. T. Oden ca 1980, on suggestion by J.-L. Lions
- **B**abuška — [Babuška 1971-73]
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Equivalence for any domain Ω:

$$\beta(\Omega) > 0 \iff \text{Lions' lemma} \iff \text{Babuška-Aziz inequality}$$

This condition (and its discrete counterpart) is called inf-sup condition or LBB condition, after

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- **B**rezzi — [Brezzi 1974]

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$$\text{Lions' Lemma} \iff \nabla \; : L^2_\circ(\Omega) \to H^{-1}(\Omega)^d \text{ is injective with closed range}$$
$$\iff \operatorname{div} \; : H^1_0(\Omega)^d \to L^2_\circ(\Omega) \text{ is surjective}$$

Babuška-Aziz inequality [Babuška-Aziz 1971], named by [Horgan-Payne 1983]

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Theorem [Bogovskiĭ 1979], [Galdi 1994]

Let $\Omega \subset \mathbb{R}^n$ be contained in a ball of radius R, starshaped with respect to a concentric ball of radius ρ. There exists a constant γ_d only depending on the dimension d such that

$$\beta(\Omega) \geq \gamma_d \left(\frac{\rho}{R}\right)^{d+1}$$

In dimension $d = 2$, we can prove

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M. Costabel, M. Dauge: On the Inequalities of Babuška-Aziz, Friedrichs and Horgan-Payne. arXiv 2013.

Starshaped domains

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John domains

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John domains

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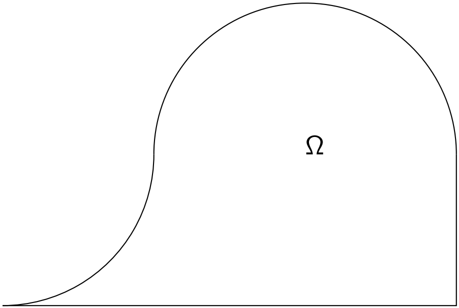

Figure: Not a John domain: Outward cusp, $\beta(\Omega) = 0$ [Friedrichs 1937]

Definition of John Domain

Definition

A domain $\Omega \subset \mathbb{R}^d$ with a distinguished point $\boldsymbol{x}_0$ is called a John domain if it satisfies the following "twisted cone" condition:
There exists a constant $\delta > 0$ such that, for any $\boldsymbol{y}$ in Ω, there is a rectifiable curve γ: $[0, \ell] \to \Omega$ parametrized by arclength such that

$$\gamma(0) = \boldsymbol{y}, \quad \gamma(\ell) = \boldsymbol{x}_0, \quad \text{and} \quad \forall t \in [0, \ell] : \ \mathrm{dist}(\gamma(t), \partial\Omega) \geq \delta\, t\,.$$

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A John domain: Union of Lipschitz domains

San Juan de la Peña, Jaca 2013

A John domain: Zigzag

Figure: A weakly Lipschitz domain: the self-similar zigzag

John domains: Spirals

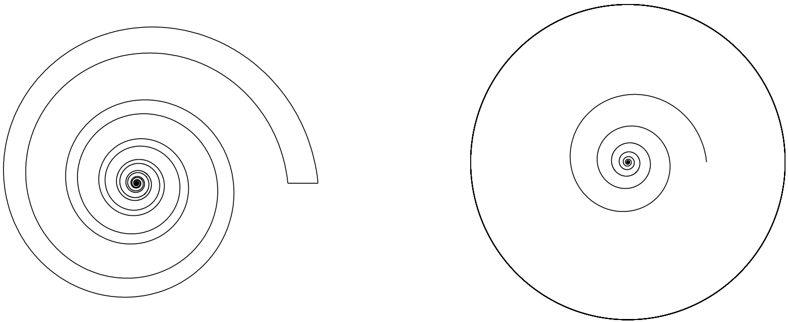


Figure: Weakly Lipschitz (left), John domain (right)

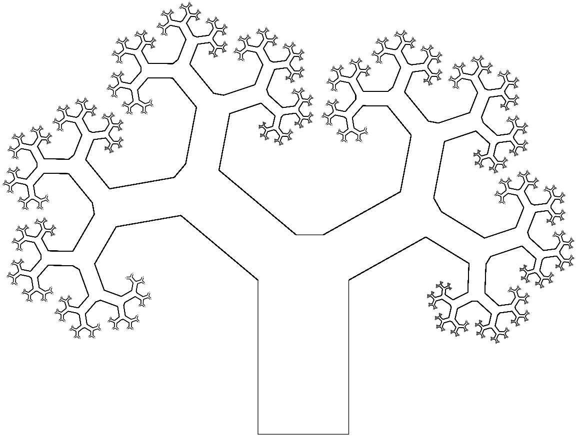
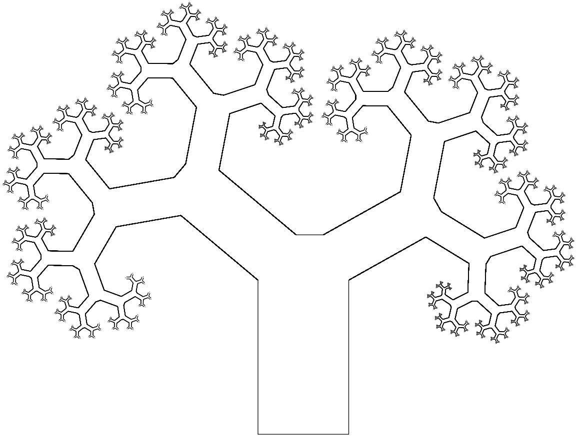


Figure: A John domain: the infinite tree

back



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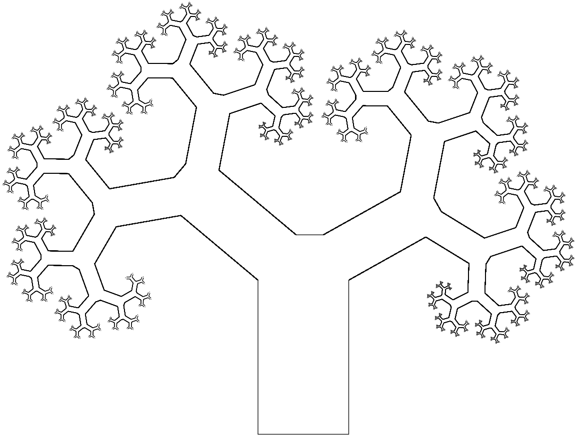
Fractal John domains: Tree or Lung



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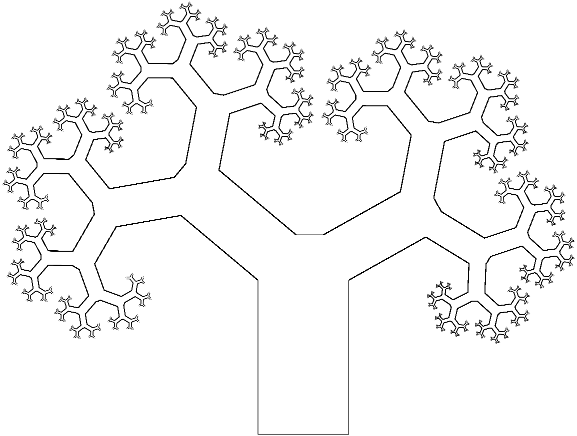

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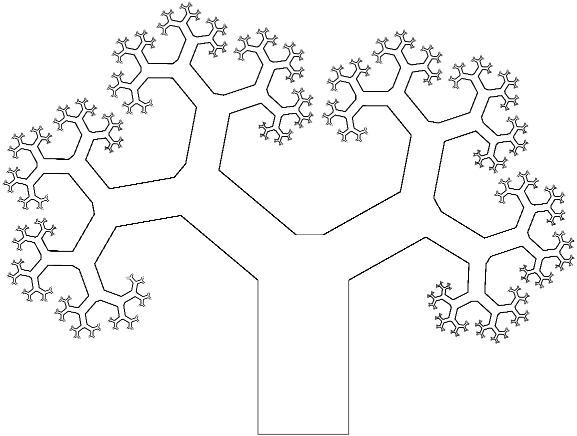

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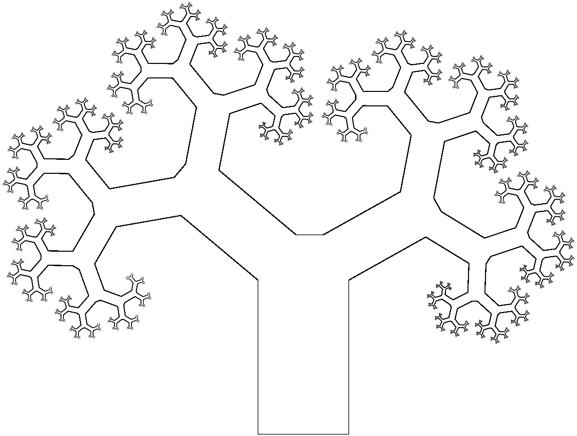

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Proof

$-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Riesz isometry.

Let $q \in L^2_\circ(\Omega)$.

$$\begin{aligned}\langle Sq, q\rangle &= \langle \operatorname{div}\Delta^{-1}\nabla q, q\rangle \\ &= \langle -\Delta^{-1}\nabla q, \nabla q\rangle \\ &= |\nabla q|_{-1}^2 \\ &= \Big(\sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{\langle \nabla q, \boldsymbol{v}\rangle}{|\boldsymbol{v}|_1}\Big)^2\end{aligned}$$

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