GAMM-Tagung 1997
Regensburg

Singularitäten an Ecken und Kanten — Was geschieht bei Parameterabhängigkeit und variablen Koeffizienten?

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Singularities

Domain $\Omega \subset \mathbb{R}^n$, boundary $\partial\Omega$ (piecewise smooth)

Linear elliptic boundary value problem

$$Lu = f \quad \text{in } \Omega \tag{1}$$

$$Bu = g \quad \text{on } \partial\Omega\,. \tag{2}$$

Abbreviation $\mathscr{L} u = \mathbf{f}$.

$$Lu = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial^\alpha u$$

Singular function u:
u does not have the (elliptic) regularity implied by the regularity of f, g and the order $2m$.

- $f \in H^{s-m}(\Omega), g \in \prod H^{s+\mu_j-1/2}(\partial\Omega), u \notin H^{s+m}(\Omega)$
- $f \in C^\infty(\overline{\Omega}), \quad g \in C^\infty(\partial\Omega), \quad u \notin C^\infty(\overline{\Omega})$

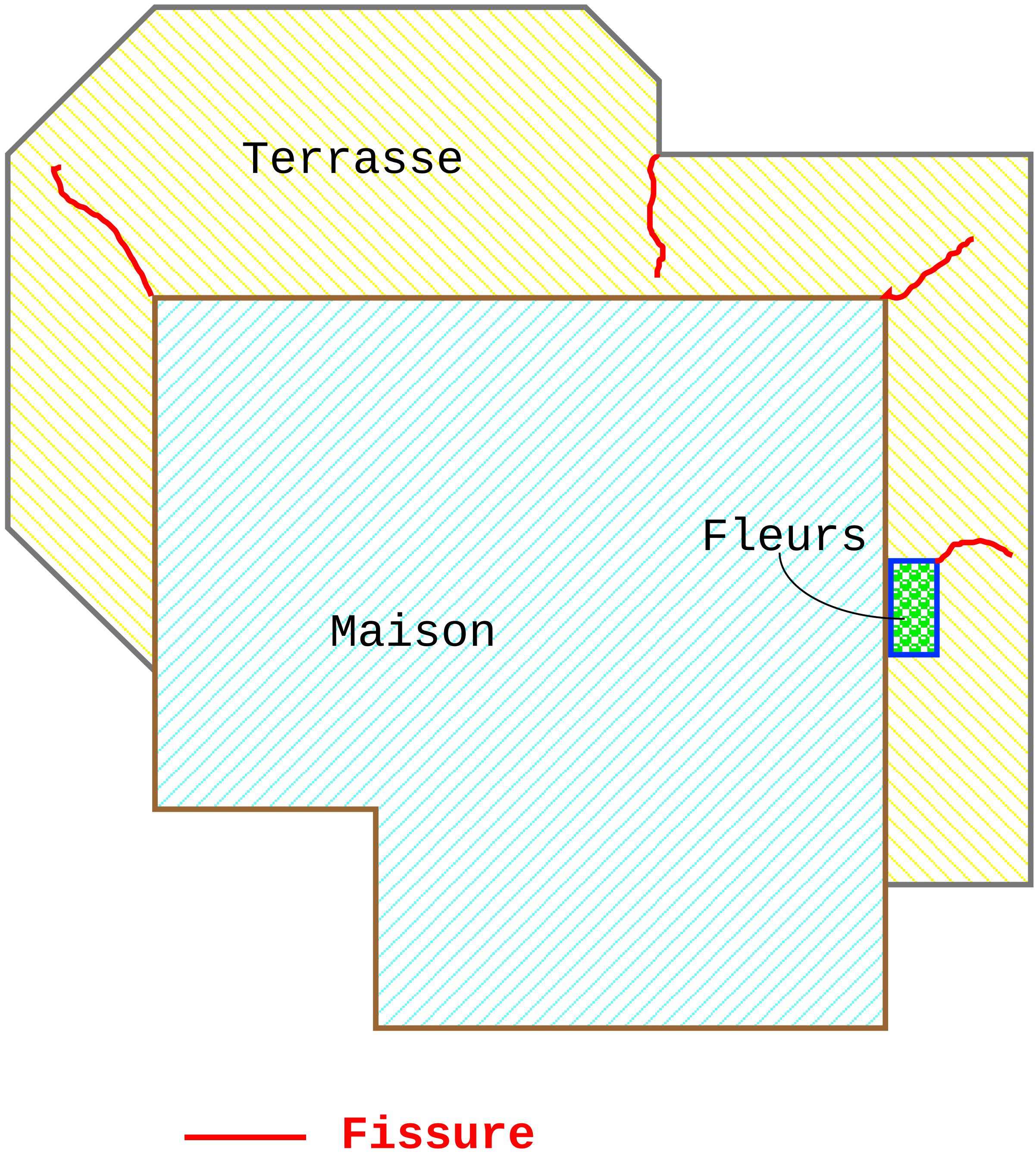
Importance: Physics (mechanical stability, stress concentration)
Numerics (bad approximation on regular grids)
Analysis (curiosity ...)

Ideal result: Set of singular functions (Σ_j), constructed from the geometry and the operators, such that

$$u = u_{\text{reg}} + \sum_j c_j \Sigma_j$$

u_{reg} has full regularity, c_j depend on $\mathbf{f}$
+ Norm estimates.

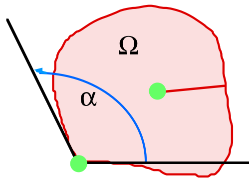
Maison et Fissures

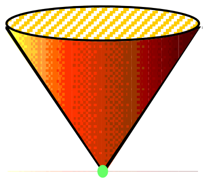


Progress in understanding corner singularities

2D corners

Dirichlet problem (Wasow, Lehman '57)
...General ell. (ADN) (Costabel&Dauge '92)

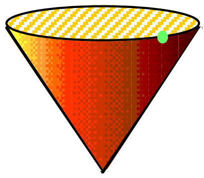




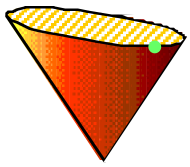
Conical points in $\mathbb{R}^n$ Regular cone

Complete technique (Kondrat'ev '67)

3D edges

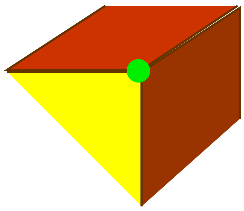


$\sim$ 2D singularities with parameter
(Kondrat'ev, Nikishkin, Maz'ya&Plamenevskij '77)
General ADN ell. (Maz'ya&Rossmann '87)



3D polyhedral corners

Straight edges meet in corner (Dauge '87)

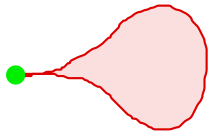


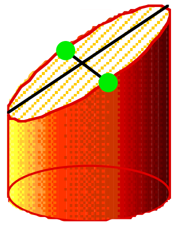
Progress (2)

Cusps Angle 0 (Feigin '72)

(Maz'ya&Plamenevskij '77)

(Steux, Dauge '87–'96)





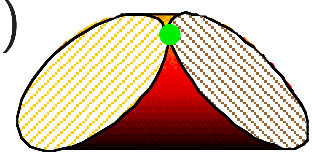
Edges in general

"Branching" and "crossing" of asymptotics

(Co&Da, Maz'ya&Rossmann, Schmutzler, Schulze '92)

General piecewise smooth domains

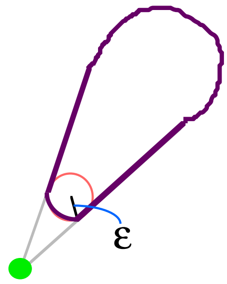
(?? 2002 ?)



Parameter-dependent problems

Singularly perturbed domains

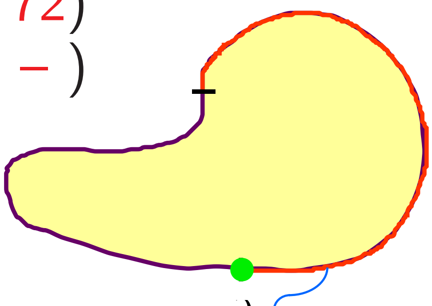
(Maz'ya&Nazarov&Plamenevskij '79)



singular perturbation in
boundary value problem

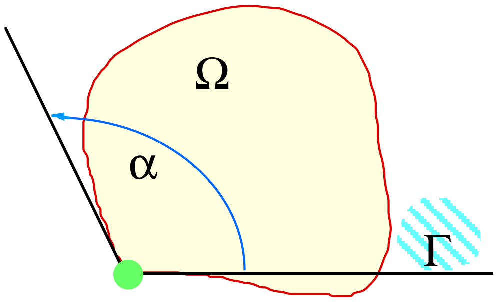
(Colli-Franzone '72)

(...Costabel&Dauge '95 –)



2D corner singularities: Dirichlet problem

$$\Delta u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$



Polar coordinates (r,θ)

Decomposition Theorem

There exist $S_j \in H^1(\Omega)$, $j \in \mathbb{N}$:

$$\left.\begin{array}{l} s \notin \{k\pi/\alpha \mid k \in \mathbb{N}\} \cup \mathbb{N} \\ f \in H^{s-1}(\Omega)\ \&\ g \in H^{s+1/2}(\partial\Omega) \end{array}\right\} \Rightarrow$$

$$\boxed{u = u_{\text{reg}} + \sum c_j S_j}\ , \quad u_{\text{reg}} \in H^{s+1}(\Omega)$$

$$\|u_{\text{reg}}\|_{s+1} + \sum |c_j| \leq C\,(\|f\|_{s-1} + \|g\|_{s+1/2})$$

Form of the singular functions:

A. Basic singularities:

$$S = r^\nu \sin\nu\theta = \mathrm{Im}(z^\nu) \quad (\nu = k\pi/\alpha,\ k \in \mathbb{N})$$

$\Delta S = 0$ in Γ, $S = 0$ on $\partial\Gamma$

B. Critical angles: $k\pi/\alpha = l \in \mathbb{N}$

$S = \mathrm{Im}(z^l log z)$ $\quad \Delta S = 0$ in Γ, S polynomial on $\partial\Gamma$

C. Higher order singularities: from curvature or l.o.t.

$S = r^{\nu+p}\,\varphi_{kp}(\theta)$ $\quad \Delta S$ singular

Complete singular function

$$S_j = \sum_{p,q} r^{\nu+p} \log^q \varphi_{pq}(\theta)$$

Basic singularities for the
Neumann problem: $r^\nu \cos\nu\theta \quad (\nu = \frac{k\pi}{\alpha})$

mixed Dirichlet-Neumann problem:

$$r^\nu \sin\nu\theta \quad (\nu = \frac{k\pi}{2\alpha})$$

Regular conical points (Kondrat'ev)

$\Omega \sim \Gamma$ near 0, Γ : cone with regular basis

3 levels of singular functions:
A. Basic singularities; B. Polynomial rhs; C.Singular rhs.

Basic singularities : Separation of variables

$S = \rho^\nu \varphi(\omega)$, (ρ, ω) polar coordinates at $0 \in \mathbb{R}^n$.

S is a homogeneous function,

solution of the totally homogeneous problem in Γ :

$$\left.\begin{array}{l} L_0 S = 0 \; : \; \Gamma \\ B_0 S = 0 \; : \; \partial\Gamma \end{array}\right\} \iff \mathscr{L}_0 S = 0$$

L_0 : principal part of L at 0: $L_0 u = \sum_{|\alpha|=2m} a_\alpha(0) \partial^\alpha u$.

Homogeneity: $\boxed{\mathscr{L}_0(\rho^\nu \varphi) = \rho^{\nu-2m} \mathscr{L}(\nu) \varphi}$

$\mathscr{L}(\nu)$: boundary value problem on $\Gamma \cap \mathbb{S}^{n-1} \subset \mathbb{S}^{n-1}$

ν singular exponent $\iff$ $\mathscr{L}(\nu)$ not invertible.

In 2D, $\mathscr{L}(\nu)$ is a Sturm-Liouville problem on $[0, \alpha]$.
2D Laplace: $\mathscr{L}(\nu)\varphi = (\varphi'' - \nu^2 \varphi, \varphi|_{\{0,\alpha\}})$

Functional analytic tool: Mellin transformation

$$\hat{u}(\lambda, \omega) = \int_0^\infty \rho^{-\lambda-1} u(\rho, \omega)\, d\rho$$

2D corners: ADN-elliptic boundary value problems

Examples: **Stokes, (anisotropic) elasticity**

$$(Lu)_k = \sum_{l=1}^{N} L_{kl} u_l, \ \mathrm{ord}\, L_{kl} \le \sigma_k - \tau_l, \ 2m = \sum_{l=1}^{N} (\sigma_l - \tau_l)$$

Multi-homogeneous functions $S_l = r^{\nu-\tau_l} \varphi_l(\theta)$

Problem Although L_0 has constant coefficients, the Sturm-Liouville operator $L(\nu)$ on $[0, \alpha]$ has variable coefficients, in general.

Answer One finds always a basis of solutions constructed from functions $(z + a\bar{z})^{\nu-\tau_l}$, where a is given by the coefficients of L_0.

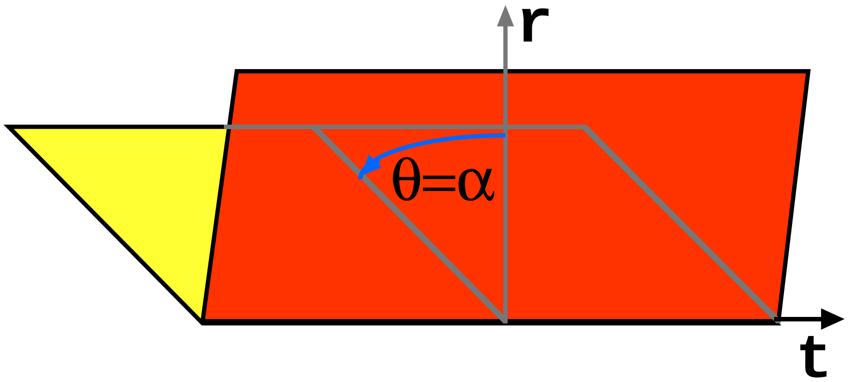
Result The basic singular functions are always combinations of functions
$S = r^\lambda \log^q r\, \varphi(\theta)$ with $\varphi(\theta) = \theta^k \cos(a\theta + b)$.

Method for computing exponents ν and singular functions S : $\boxed{\det A(\nu) = 0}$

$A(\nu)$: matrix $(2m \times 2m)$ of boundary conditions on $\partial\Gamma$ of a basis of solutions of $L(\nu)\varphi = 0$

- Simple and explicit, even for anisotropic elasticity and interface problems with several different materials.

3D edges: 2D corners with a parameter



Constant angle α and constant coefficients in t:

Singularity $c(t)S(r,\theta)$, $c(t)$ "regular".

Example: 3D Δ crack (with smooth crack front):

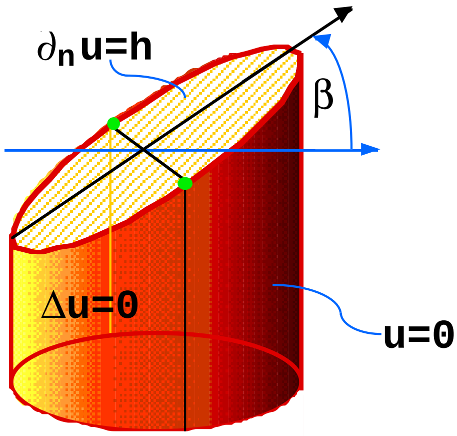
$\alpha = 2\pi$, $S(r,\theta) = r^{1/2}\cos\theta/2$

$c(t)$: Stress Intensity Factor

Problems if

$\alpha = \alpha(t)$ and/or the operator $\mathscr{L}$ depends on t

The Skew Cylinder



$$\alpha(t) \in [\frac{\pi}{2} - \beta, \frac{\pi}{2} + \beta]$$
$$\nu_1(t) = \pi/(2\alpha(t))$$
$$\alpha(t_{0,1}) = \pi/2$$

$$S_1(t,r,\theta) = \begin{cases} r^{\nu_1(t)} \cos \nu_1(t)\theta & : \alpha \neq \frac{\pi}{2} \\ r(\log r \cos\theta - \theta \sin\theta) & : t_0, t_1 \end{cases}$$

Coefficient $c_1(t) = b_1(t) - \dfrac{h(t,0)}{\cos\alpha(t)}$, $b_1(t)$ regular

Explanation: Do not forget $\nu_2 = 1$:

$S_2(t) = r\sin(\alpha(t) - \theta)$ smooth Taylor term

$c_2(t) = h(t,0)/\cos\alpha(t)$

Crossing $\nu_1 = \nu_2$

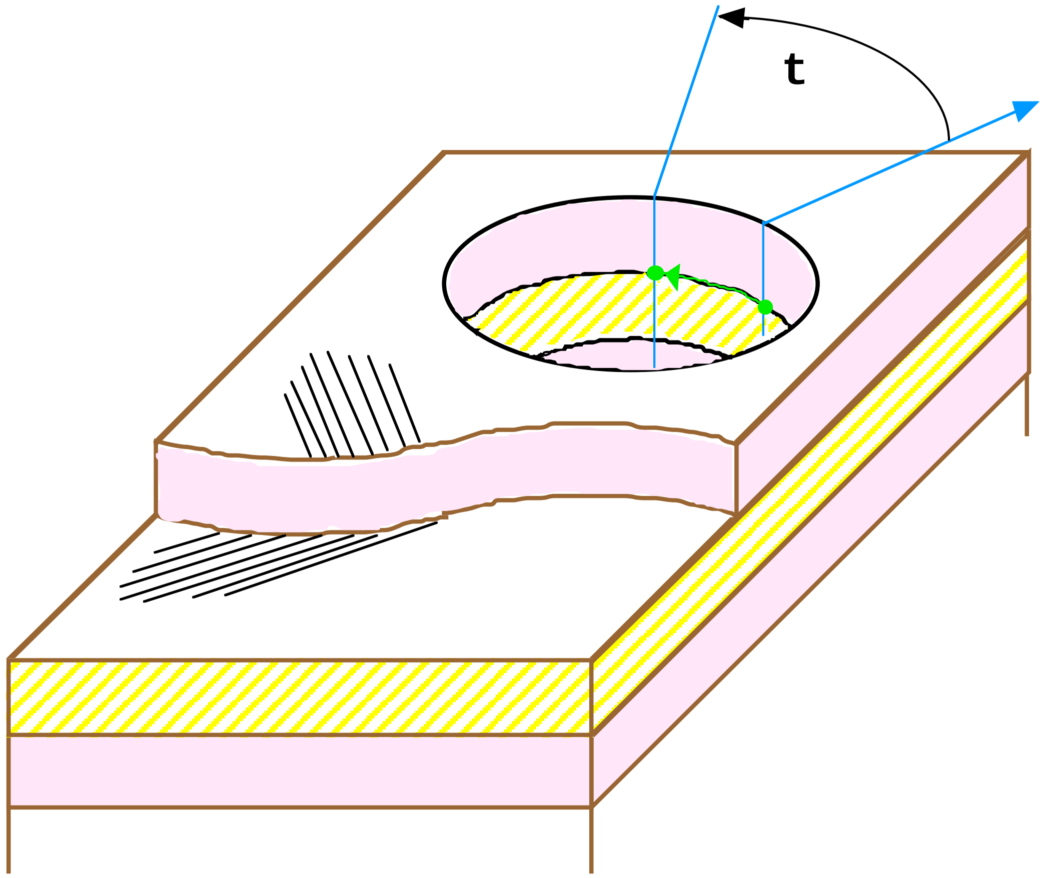
Basis

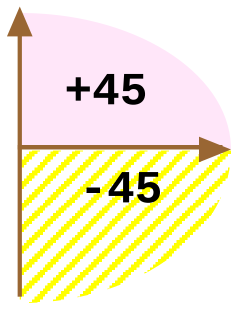
$$X_1(t,r,\theta) = r^{\nu_1(t)} \cos \nu_1(t)\theta = S_1(t,r,\theta)$$
$$X_2 = (S_1 - S_2)/(\nu_1(t) - \nu_2(t))$$

regular basis of $\operatorname{span}\{S_1(t), S_2(t)\}$

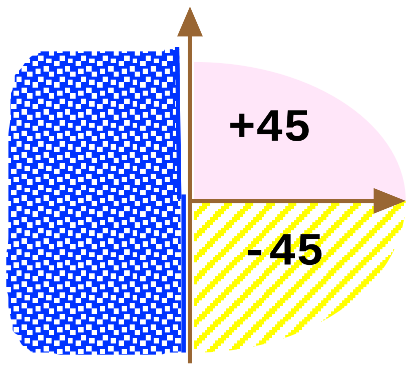
$$c_1 S_1 + c_2 S_2 \equiv b_1 X_1 + b_2 X_2 \qquad b_1, b_2 \text{ regular}$$

Bolt hole in laminated material



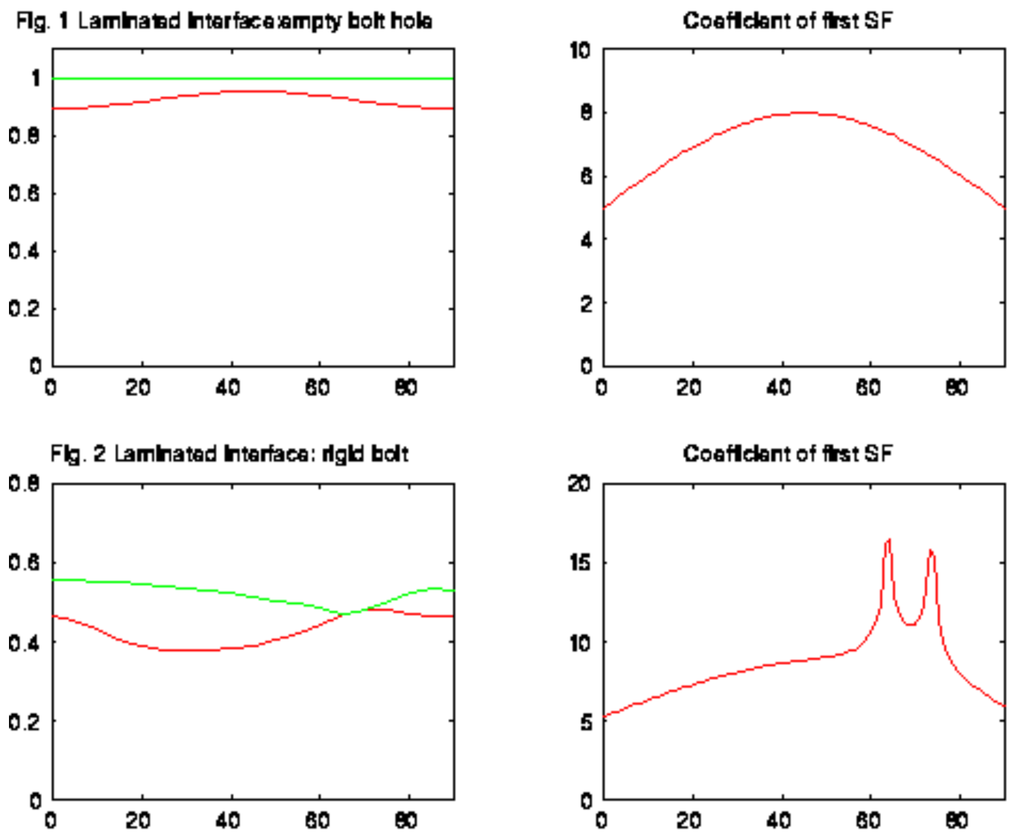


empty hole



rigid bolt in hole
(semi-detached)

Bolt hole in laminated material (2)



Branching of exponents $\nu_1 = \nu_2$ in t_0, t_1

$\nu_1(t) = \overline{\nu_2(t)}$ complex for $t_0 < t < t_1$.

General recipe: Divided differences

$$S[p,q;r] = \frac{1}{2\pi i}\int_{\mathscr{C}} r^{\lambda}\,\frac{q(\lambda)}{p(\lambda)}\,d\lambda, \qquad p,q \text{ polynomial}$$

Exemple: $p = \lambda^2 - t$

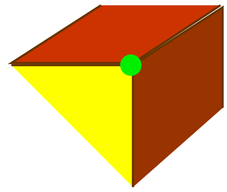
$q = 2:\ S_1 = (r^{\sqrt{t}} - r^{-\sqrt{t}})/\sqrt{t}$

$q = 2\lambda:\ S_2 = r^{\sqrt{t}} + r^{-\sqrt{t}}$

Divided differences: $S[\nu_1,\ldots,\nu_k] = \dfrac{1}{2\pi i}\displaystyle\int_{\mathscr{C}} \frac{S(\lambda)}{\prod(\lambda-\nu_j)}\,d\lambda$

3D polyhedral corner (M. Dauge)

Cone with polygonal basis



$\Sigma(\rho,\omega) = \rho^\lambda \psi(\omega):$ ψ has 2D corner singularities

Edges meet in corner

$S(\rho,r,\theta) = \gamma(\rho) r^\nu \varphi(\theta):$ $\gamma(\rho)$ is singular in $\rho = 0$.

2-step decomposition

1. Corner decomposition

$$u = u_{\text{reg}}^{\text{corner}} + \sum c_j \Sigma_j$$

$u = u_{\text{reg}}^{\text{corner}}$ is not regular, but flat

Dirichlet problem $\Delta u = f \in H^{s-1}(\Omega), u \in H_0^1(\Omega)$

$u_{\text{reg}}^{\text{corner}} \in H^1$, $\rho^{-s}(\rho\partial_\rho)^k u_{\text{reg}}^{\text{corner}} \in L^2 \quad (k \le m)$

2. Edge decomposition

$$u_{\text{reg}}^{\text{corner}} = u_{\text{reg}} + \sum_{\text{edges}} \sum_{\nu pq} \tilde{\gamma}_{\nu p}(\rho)\, r^{\nu+p} \log^q r\, \varphi_{pq}(\rho)$$

$u_{\text{reg}} \in H^{s+1}$, $\tilde{\gamma}_{\nu p}(\rho) = O(\rho^{s-3/2})$ in 0.

Total edge coefficient $\gamma(\rho) = \sum_{\lambda q} a_{\lambda q} \rho^\lambda \log^q \rho + \tilde{\gamma}(\rho)$

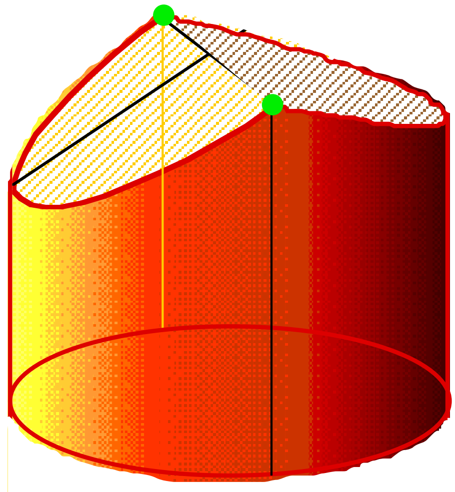
[Q] Which is the main singularity, S_1 (e) or Σ_1 (c)?

[A] That depends: S_1 is 2D, Σ_1 is 3D.

Fichera's corner: $\nabla u \in L^{6000/1133}$, $u \in H^{5/3-\varepsilon}$

3D corners and curved edges

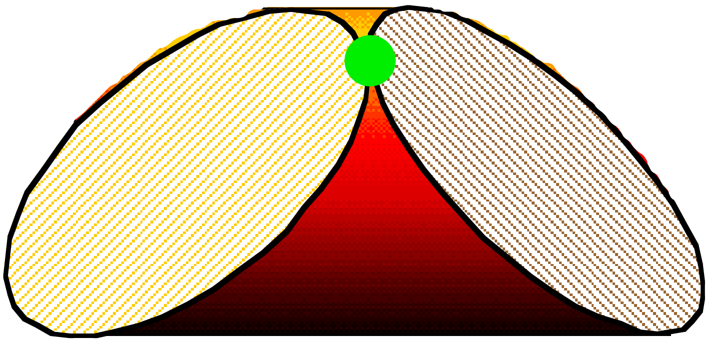
Exemple 1.



Solved: Divided differences + 2-step decomposition

(Costabel&Dauge '95)

Exemple 2.



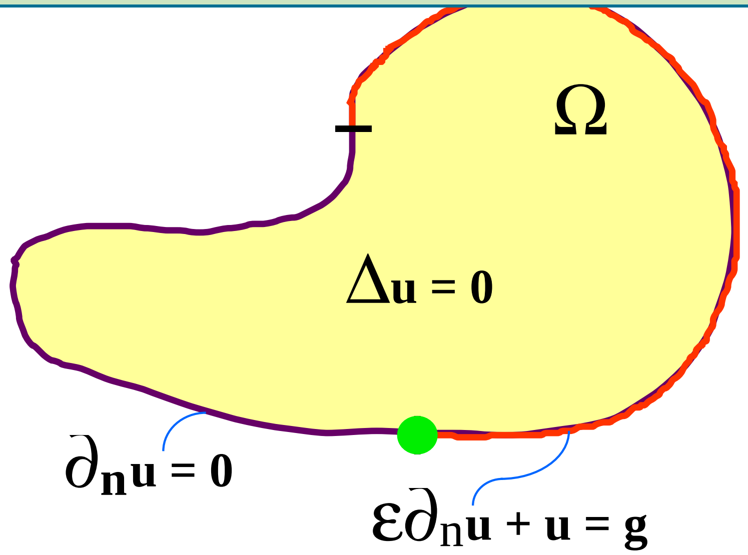
Unsolved: Edges are tangential in corner

Condition for tangential planes T_j in corner c

$j \neq k \Longrightarrow T_j \neq T_k$

j, k, l distinct $\Longrightarrow T_j \cap T_k \cap T_l = \{c\}$

A singular singular perturbation problem



Convergence $u_\varepsilon \to u_0$ in $H^{1+\delta}(\Omega)$ (Colli-Franzone '72)

$\varepsilon > 0$ Neumann / Neumann + lower order
leading singularity $\gamma(\varepsilon)\, r\, (\log r \cos\theta + (\pi - \theta)\sin\theta)$

$\varepsilon = 0$ Neumann / Dirichlet
leading singularity $c_1\, r^{1/2} \sin\theta/2$

Complete asymptotic expansion (Costabel&Dauge '95)

$$u_\varepsilon = \sum_{n\geq 0} \varepsilon^n \Big(u^n[\log\varepsilon](x) + \sqrt{\varepsilon}\, w^n[\log\varepsilon](\tfrac{x}{\varepsilon}) \Big)$$

$u^0 = u_0\,,\ u^n$ singular as u_0
w^n corner layers: at 0 singular as u_ε, at ∞ $O(r^{-1/2})$

[Q1] Behavior of $\gamma(\varepsilon)$ as $\varepsilon \to 0$?

[Q2] Origin of $r^{1/2} \sin\theta/2$ as $\varepsilon \to 0$?

[A] Existence of a function

K_1 : at ∞ $\sim r^{1/2} \sin\theta/2$
at 0 $\sim r\,(\log r \cos\theta + (\pi-\theta)\sin\theta)$
$u_\varepsilon \sim u_0 + \sqrt{\varepsilon}\, K_1(r/\varepsilon, \theta)\,.$