ON THE LIMIT SOBOLEV REGULARITY FOR DIRICHLET AND NEUMANN PROBLEMS ON LIPSCHITZ DOMAINS

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ABSTRACT. We construct a bounded C^1 domain Ω in $\mathbb{R}^n$ for which the $H^{3/2}$ regularity for the Dirichlet and Neumann problems for the Laplacian cannot be improved, that is, there exists f in $C^\infty(\overline{\Omega})$ such that the solution of $\Delta u = f$ in Ω and either $u = 0$ on $\partial\Omega$ or $\partial_n u = 0$ on $\partial\Omega$ is contained in $H^{3/2}(\Omega)$ but not in $H^{3/2+\varepsilon}(\Omega)$ for any $\varepsilon > 0$. An analogous result holds for L^p Sobolev spaces with $p \in (1, \infty)$.

1. INTRODUCTION

The motivation for this note comes from a question of regularity of the time-harmonic Maxwell equations in Lipschitz domains. In the variational theory of Maxwell's equations, basis for the analysis of many algorithms of numerical electrodynamics, the following two function spaces are fundamental:

$$\begin{aligned} X_N &= H(\operatorname{div}, \Omega) \cap H_0(\operatorname{curl}, \Omega) \\ &= \{u \in L^2(\Omega; \mathbb{C}^3) \mid \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega; \mathbb{C}^3), u \times n = 0 \text{ on } \partial\Omega\} \qquad (1.1) \\ X_T &= H_0(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega) \\ &= \{u \in L^2(\Omega; \mathbb{C}^3) \mid \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega; \mathbb{C}^3), u \cdot n = 0 \text{ on } \partial\Omega\} \qquad (1.2) \end{aligned}$$

Here n is the outward unit normal vector field on the boundary of the domain $\Omega \subset \mathbb{R}^3$.

If Ω is a bounded Lipschitz domain, then it has been known for a long time [15, 11] that X_N and X_T are compactly embedded subspaces of $L^2(\Omega; \mathbb{C}^3)$, and it has been shown more precisely [5, 10] that they are contained in the Sobolev space $H^{\frac{1}{2}}(\Omega, \mathbb{C}^3) = W^{\frac{1}{2},2}(\Omega, \mathbb{C}^3)$. For large classes of more regular domains, X_N and X_T are contained in $H^1(\Omega, \mathbb{C}^3)$ (see [3] for $C^{1,1}$ domains, [7] for $C^{\frac{3}{2}+\varepsilon}$ domains, [12] for X_N on convex domains, [13] for "almost convex" domains). The regularity is diminished by corner singularities, but one also knows [3] that for every Lipschitz polyhedron or, more generally, piecewise smooth domain Ω that is at least C^2-diffeomorphic to a polyhedron, there exists $\varepsilon > 0$ such that

$$X_N \cup X_T \subset H^{\frac{1}{2}+\varepsilon}(\Omega; \mathbb{C}^3)\,. \qquad (1.3)$$

The additional regularity described by ε is of some use in the numerical analysis of Maxwell's equations (see for example [2, 1]). The parameter ε can become arbitrarily small, depending on the corner angles of $\partial\Omega$, but it depends only on these angles, that is, on the local Lipschitz constant of $\partial\Omega$. Based on this observation, one could ask the question whether for any Lipschitz domain Ω, there exists such an $\varepsilon > 0$ for which (1.3) holds. This question is the motivation for the present investigation.

Date: 2018-12-04.

To the best of the author's knowledge, the conjecture that such an $\varepsilon > 0$ always exists is not incompatible with the currently available regularity results for Maxwell's equations on Lipschitz domains, but we shall show that it is not true. As a corollary of our constructions, we obtain a counterexample that is even C^1.

Proposition 1.1. *There exists a bounded C^1 domain $\Omega \subset \mathbb{R}^3$, an $L^2(\Omega)$ function g and an $L^2(\Omega;\mathbb{C}^3)$ function h such that the solutions $u \in L^2(\Omega;\mathbb{C}^3)$ of the system*

$$\operatorname{div} u = g\,, \qquad \operatorname{curl} u = h \quad \text{in } \Omega \tag{1.4}$$

and either

$$u \times n = 0 \quad \text{on } \partial\Omega \tag{1.5}$$

or

$$u \cdot n = 0 \quad \text{on } \partial\Omega \tag{1.6}$$

do not belong to $H^{\frac{1}{2}+\varepsilon}(\Omega;\mathbb{C}^3)$ for any $\varepsilon > 0$.
In the system (1.4), the field h can be chosen to be zero and g can be chosen to be continous on $\overline{\Omega}$.

As we will see in the following, analogous results are true in dimension 2 and in higher dimensions, and also for non-Hilbert Sobolev spaces over L^p with p different from 2.

Non-regular solutions of the div-curl system (1.4) are typically sought as gradients of solutions of the inhomogeneous Laplace (Poisson) equation with either Dirichlet (for (1.5)) or Neumann (for (1.6)) boundary conditions. A non-regularity result for these Laplace boundary value problems is the main result of this paper, see Theorem 1.2 below. It will be proved in Section 3 for dimension $d = 2$ and in Section 4 for higher dimensions.

We use the standard notation $W^{s,p}(\Omega)$ for the Sobolev-Slobodeckij spaces on $\Omega \subset \mathbb{R}^d$, and we recall that for $0 < s < 1$ the seminorm

$$|u|_{s,p;\Omega} = \left(\int_\Omega \int_\Omega \frac{|u(y) - u(x)|^p}{|y - x|^{d+sp}} dx\, dy \right)^{\frac{1}{p}} \tag{1.7}$$

defines the norm $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + |u|_{s,p;\Omega}$, that $W^{0,p}(\Omega) = L^p(\Omega)$, and that for any s there holds

$$u \in W^{s+1,p}(\Omega) \iff u \in W^{s,p}(\Omega) \text{ and } \nabla u \in W^{s,p}(\Omega;\mathbb{C}^d)\,.$$

As usual, we write $W^{s,p}_0(\Omega)$ for the closure of $C^\infty_0(\Omega)$ in $W^{s,p}(\Omega)$. In view of an interesting property of the domain we are going to construct (see equations (1.10) and (4.1)), we recall that for $\frac{1}{p} < s < 1 + \frac{1}{p}$ the subspace $W^{s,p}_0(\Omega)$ is characterized by the condition that the boundary trace vanishes, whereas for $1 + \frac{1}{p} < s < 2 + \frac{1}{p}$ the condition is that both the trace and the normal derivative vanish on $\partial\Omega$.

In order to describe known regularity results, we also need the Bessel potential spaces $H^{s,p}(\Omega)$, which are different from $W^{s,p}(\Omega)$ if $p \neq 2$. For the main properties of these spaces, see [14]. In Triebel's notation $W^{m,p}(\Omega) = F^m_{p,2}(\Omega)$ for $m \in \mathbb{N}$ and

$$H^{s,p}(\Omega) = F^s_{p,2}(\Omega)\,, \qquad \text{and for } s \notin \mathbb{Z}: \; W^{s,p}(\Omega) = B^s_{p,p}(\Omega)\,.$$

Note that the trace space for both $W^{s,p}(\Omega)$ and $H^{s,p}(\Omega)$ on a sufficiently smooth boundary is $W^{s-\frac{1}{p},p}(\partial\Omega)$ if $s > \frac{1}{p}$.

Comprehensive regularity results in the $H^{s,p}$ spaces for the Dirichlet problem on Lipschitz domains were given by Jerison and Kenig [8]. They had previously studied the homogeneous Laplace equation with inhomogeneous Neumann conditions [9], and corresponding results for the homogeneous Neumann problem of the inhomogeneous Laplace equation were obtained by Fabes, Mendez and Mitrea [6] and Zanger [16]. In particular, there exist precise answers to the question for which s and p the condition $g \in H^{s-2,p}(\Omega)$ implies $v \in H^{s,p}(\Omega)$ for the solutions v of the problems

$$\Delta v = g \quad \text{in } \Omega\,, \qquad\qquad v = 0 \quad \text{on } \partial\Omega \tag{1.8}$$

$$\Delta v = g \quad \text{in } \Omega\,, \qquad\qquad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.9}$$

For the maximal regularity one finds a limit at $s = 1 + \frac{1}{p}$. We summarize the main results pertaining to the question of maximal regularity (here formulated for the Dirichlet problem, see [8, Thms 1.1–1.3], where $H^{s,p}$ is written L^p_s; the results for the Neumann problem are similar):

For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, there exists $p_0 \in [1, 2)$ such that for $p_0 < p < \frac{p_0}{p_0-1}$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$ the solution v of the Dirichlet problem (1.8) with $g \in H^{s-2,p}(\Omega)$ belongs to $H^{s,p}(\Omega)$. The following 4 points indicate the known borders of this result.
1. In general, $p_0 > 1$ and there are counterexamples as soon as p or s are outside of the given bounds, but when Ω is a C^1 domain, one can choose $p_0 = 1$, so that the result is true for any $p \in (1, \infty)$.
2. When $p > 2$, there are Lipschitz counterexamples with $g \in C^\infty(\overline{\Omega})$ and $v \notin W^{1+\frac{1}{p},p}(\Omega)$. There is a C^1 counterexample for $p = 1$ with $g \in C^\infty(\overline{\Omega})$ and $v \notin W^{2,1}(\Omega)$.
3. In the optimal regularity-shift result for C^1 domains, the condition on s cannot be weakened, because for any $p > 1$ there exists a bounded C^1 domain Ω and a $g \in H^{-1+\frac{1}{p},p}(\Omega)$ such that $v \notin H^{1+\frac{1}{p},p}(\Omega)$.
4. On the other hand, if g is more regular, for example $g \in H^{-1+\frac{1}{p}+\varepsilon,p}(\Omega)$ for some $\varepsilon > 0$ and $p > 1$, then $v \in H^{1+\frac{1}{p},p}(\Omega)$ follows. The latter result is obtained by subtracting from v a solution $v_0 \in H^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ of $\Delta v_0 = g$ without boundary conditions and observing that a harmonic function with trace in $W^{1,p}(\partial\Omega)$ belongs to $H^{1+\frac{1}{p},p}(\Omega)$.

We will prove that one will have $v \notin H^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ for any $\varepsilon > 0$, in general, even for more regular g. Because of the mutual inclusions $H^{s+\varepsilon,p} \subset W^{s,p} \subset H^{s-\varepsilon,p}$ for any $\varepsilon > 0$, the result is equivalently formulated in the scale of $W^{s,p}$ spaces.

Theorem 1.2. *In $\mathbb{R}^d$, $d \geq 2$, there exists a bounded C^1 domain Ω and for both the Dirichlet problem* (1.8) *and the Neumann problem* (1.9) *functions $g \in L^\infty(\Omega)$ such that the solutions $v \in H^1(\Omega)$ do not belong to $W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ for any $p \in [1, \infty)$ and any $\varepsilon > 0$.*

Remark 1.3. It will follow from the proof that in dimension $d = 2$, there are functions $g \in C^\infty(\overline{\Omega})$ that provide examples, even $g = 1$ is possible for the Dirichlet problem and a second degree polynomial g for the Neumann problem. See also Remark 3.3. In dimension $d \geq 3$, there is still an example with $g = 1$ for the Dirichlet problem, and examples with $g \in C^\alpha(\overline{\Omega})$, $\alpha > 0$, for the Neumann problem.

Remark 1.4. Not all of this is new: For $p = 1$, the counterexample from [8, Theorem 1.2(b)] shows that the result for the Dirichlet problem holds even with $\varepsilon = 0$. Moreover, for $p > 2$ the result of Theorem 1.2 is not interesting in the class of Lipschitz domains, because singularities at conical points provide a limit of regularity that is strictly below $s = 1 + \frac{1}{p}$. This follows from the well-known singular asymptotic behavior $O(r^\alpha)$ near a straight $d-2$ dimensional facet of the boundary (corner in dimension $d = 2$ or "edge" in dimension ≥ 3) of opening angle $\frac{\pi}{\alpha}$ of generic solutions of the Dirichlet and Neumann problems with smooth right hand sides, where r is the distance to the corner or edge. Such functions are in $W^{s,p}(\Omega)$ only for $s < \alpha + \frac{2}{p}$, hence not in $W^{1+\frac{1}{p},p}(\Omega)$ as soon as the opening angle exceeds $\frac{p}{p-1}\pi$. But for C^1 domains the result still seems to be new even for $p > 2$. We provide a proof that works for any $p \geq 1$, because there is no extra cost with respect to the proof for $p = 2$. One just has to be careful to observe that the same domain Ω and the same function g give an example valid for all p and all ε.

Proposition 1.1 follows from Theorem 1.2 for $p = 2$, $d = 3$ if we take $u = \nabla v$ ("electrostatic field"). The Laplace equation for v implies the div-curl system (1.4) for u with $h = 0$, and the Dirichlet and Neumann conditions in (1.8) and (1.9) for v imply the vanishing of the tangential component (1.5) or of the normal component (1.6), respectively. Finally, $v \in W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ is equivalent to $u \in W^{\frac{1}{p}+\varepsilon,p}(\Omega;\mathbb{C}^3)$.

The construction of our counterexample uses the ideas of Filonov in the paper [7], where he considers a related question for $\varepsilon = \frac{1}{2}$ and constructs a $C^{\frac{3}{2}}$ domain Ω that satisfies, among other interesting properties

$$H^2(\Omega) \cap H_0^1(\Omega) = H_0^2(\Omega)\,,$$

that is, the homogeneous Dirichlet condition for H^2 functions implies the homogeneous Neumann condition, see also [4]. Generalizing this, the C^1 domain Ω that we will construct satisfies

$$W^{1+\frac{1}{p}+\varepsilon,p}(\Omega) \cap W_0^{1,p}(\Omega) = W_0^{1+\frac{1}{p}+\varepsilon,p}(\Omega) \quad \forall 1 \leq p < \infty\,, 0 < \varepsilon < 1\,. \tag{1.10}$$

2. Generalizing Filonov's separating function

We construct a continuous real-valued function f on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with the following property: If a and b belong to $W^{\varepsilon,p}(\mathbb{T})$ for some $\varepsilon > 0$, $p \geq 1$, and $af = b$, then $a = b = 0$.

The construction and proof are modeled after Filonov's construction of a $C^{\frac{1}{2}}$ function that has the above separation property for $\varepsilon = \frac{1}{2}$ and $p = 2$. It is in the lineage of Weierstrass' example of a continuous nowhere differentiable function.

We define f via a lacunary Fourier series

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(b_k x) = \sum_{k=1}^{\infty} f_k(x) \tag{2.1}$$

where the sequences $a_k > 0$ and $b_k \in \mathbb{N}$ are chosen so that they satisfy $\sum a_k < \infty$ and $b_k \geq 2$, $b_{k+1} \geq 2b_k$, $k \geq 1$, and the following properties for a given small constant $\gamma > 0$ to be fixed later on (see (2.7)):

$$\sum_{k=1}^{m-1} a_k b_k \le \gamma\, a_m b_m \qquad \forall\, m \ge 2 \tag{2.2}$$

$$\sum_{k=m+1}^{\infty} a_k \le \gamma\, a_m \qquad \forall\, m \ge 1 \tag{2.3}$$

$$\sum_{m=1}^{\infty} a_m^p b_m^{p\varepsilon} = +\infty \qquad \forall\, \varepsilon > 0,\ p \ge 1\,. \tag{2.4}$$

We first show that for sufficiently large $q \in \mathbb{N}$ the sequences $a_k = q^{-k}$, $b_k = 2^{q^k}$ have the properties (2.2)–(2.4), and we shall keep this choice from now on [1] .

For (2.2), let $s_m = \frac{1}{a_m b_m}\sum_{k=1}^{m-1} a_k b_k$. Noting that for $q \ge 7$ we have $q^2\, 2^{1-q} < 1$, we show by induction that then $s_m < \frac{1}{q-1}$ for all $m \ge 2$, which implies (2.2) for q large enough. Indeed,

$$s_2 = \tfrac{a_1 b_1}{a_2 b_2} = q\, 2^{(1-q)q} < q\, 2^{1-q} < \tfrac{1}{q} < \tfrac{1}{q-1}\,,$$

and if $s_m < \frac{1}{q-1}$ it follows that

$$s_{m+1} = (s_m + 1)\tfrac{a_m b_m}{a_{m+1} b_{m+1}} = (s_m+1)\, q\, 2^{(1-q)q^m} < (s_m+1)\, q\, 2^{(1-q)} < (\tfrac{1}{q-1}+1)\tfrac{1}{q} = \tfrac{1}{q-1}\,.$$

For (2.3), we have

$$\sum_{k=m+1}^{\infty} \frac{a_k}{a_m} = \sum_{k=1}^{\infty} q^{-k} = \frac{1}{q-1}$$

which again is less than γ for q large enough.

For (2.4) we use that $2^t \ge t\log 2$ for all $t > 0$, so that $a_m^p b_m^{p\varepsilon} = (2^{\varepsilon q^m}/q^m)^p \ge (\varepsilon \log 2)^p$ for all m.

Lemma 2.1. *The function f defined by* (2.1) *is continuous on $\mathbb{T}$ and satisfies*

$$\int_0^{2\pi} \frac{|f(y)-f(x)|^p}{|y-x|^{1+p\varepsilon}}dy = +\infty \qquad \textit{for all } x \in [0,2\pi],\ \varepsilon > 0, 1 \le p < \infty\,. \tag{2.5}$$

Proof. We first note that we have $f(2\pi - x) = -f(x)$, so that it is sufficient to prove (2.5) for $x \in [0,\pi]$. In this case $[x, x+1] \subset [0, 2\pi]$, and therefore with the disjoint intervals $I_m = [\frac{1}{b_m}, \frac{2}{b_m}]$ we have

$$\int_0^{2\pi} \frac{|f(y)-f(x)|^p}{|y-x|^{1+p\varepsilon}}dy \ge \sum_{m=1}^{\infty} \int_{I_m} \frac{|f(x+h)-f(x)|^p}{|h|^{1+p\varepsilon}}dh \tag{2.6}$$

Now for $h \in I_m$ we estimate

$$\Big(\int_{I_m} \frac{|f(x+h)-f(x)|^p}{|h|^{1+p\varepsilon}}dh\Big)^{\frac1p} \ge J_1 - J_2$$

with $J_1 = \Big(\int_{I_m} \frac{|f_m(x+h)-f_m(x)|^p}{|h|^{1+p\varepsilon}}dh\Big)^{\frac1p}$ and $J_2 = \sum_{k\ne m}\Big(\int_{I_m} \frac{|f_k(x+h)-f_k(x)|^p}{|h|^{1+p\varepsilon}}dh\Big)^{\frac1p}$.

[1] By plotting approximate values of the integral in (2.7) against the variable z and visual inspection of the graph, one can obtain a rough numerical approximation of γ that indicates that $\gamma \ge 0.0154$. In view of the condition $1/(q-1) < \gamma$, this suggests that a value of $q = 66$ should be "sufficiently large".

To estimate J_1, we assume that $0 < \varepsilon < 1$ and make the change of variables $t = b_m h$ to obtain

$$J_1 = a_m b_m^\varepsilon \Big(\int_1^2 |\sin(b_m x + t) - \sin(b_m x)|^p t^{-(1+p\varepsilon)} dt \Big)^{\frac{1}{p}} \ge 5\,\gamma\, a_m b_m^\varepsilon\,,$$

where we defined

$$\gamma = \tfrac{1}{5} \min_{z\in\mathbb{T}} \int_1^2 |\sin(z+t) - \sin(z)| t^{-2} dt > 0\,. \tag{2.7}$$

Here we used Hölder's inequality,

$$\begin{aligned}\int_1^2 \frac{|\sin(z+t) - \sin(z)|}{t^2} dt &\le \int_1^2 \frac{|\sin(z+t) - \sin(z)|}{t^{1+\varepsilon}} dt \\ &\le \Big(\int_1^2 |\sin(z+t) - \sin(z)|^p\, t^{-(1+p\varepsilon)} dt \Big)^{\frac{1}{p}} \Big(\int_1^2 \frac{dt}{t} \Big)^{1-\frac{1}{p}}.\end{aligned}$$

To estimate J_2, we use for $k \le m-1$

$$|f_k(x+h) - f_k(x)| \le a_k b_k |h| \le 2 a_k b_k \tfrac{1}{b_m}$$

and for $k \ge m+1$

$$|f_k(x+h) - f_k(x)| \le 2a_k$$

so that we obtain with (2.2)

$$\sum_{k=1}^{m-1} \Big(\int_{I_m} \frac{|f_k(x+h) - f_k(x)|^p}{|h|^{1+p\varepsilon}} dh \Big)^{\frac{1}{p}} \le 2\gamma a_m \Big(\int_{I_m} \frac{dh}{|h|^{1+p\varepsilon}} \Big)^{\frac{1}{p}} \le 2\gamma a_m b_m^\varepsilon$$

and with (2.3)

$$\sum_{k=m+1}^{\infty} \Big(\int_{I_m} \frac{|f_k(x+h) - f_k(x)|^p}{|h|^{1+p\varepsilon}} dh \Big)^{\frac{1}{p}} \le 2\gamma a_m \Big(\int_{I_m} \frac{dh}{|h|^{1+p\varepsilon}} \Big)^{\frac{1}{p}} \le 2\gamma a_m b_m^\varepsilon\,,$$

hence $J_2 \le 4\gamma a_m b_m^\varepsilon$.

Together, this gives

$$\Big(\int_{I_m} \frac{|f(x+h) - f(x)|^p}{|h|^{1+p\varepsilon}} dh \Big)^{\frac{1}{p}} \ge \gamma\, a_m b_m^\varepsilon\,,$$

and finally with (2.6) and (2.4)

$$\int_0^{2\pi} \frac{|f(y) - f(x)|^p}{|y-x|^{1+p\varepsilon}} dy \ge \sum_{m=1}^{\infty} \gamma^p a_m^p b_m^{p\varepsilon} = +\infty\,.$$

$\square$

Proposition 2.2. *The function f defined by* (2.1) *has the following separation property: Let $0 < \varepsilon < 1$, $p \ge 1$ and $a, b \in W^{\varepsilon,p}(0, 2\pi)$. If $af = b$, then $a = b = 0$.*

Proof. Write the $W^{\varepsilon,p}$ seminorm as in (1.7)

$$|b|_{\varepsilon,p} = \Big(\int_0^{2\pi} \int_0^{2\pi} \frac{|b(y) - b(x)|^p}{|y-x|^{1+p\varepsilon}} dy\, dx \Big)^{\frac{1}{p}}.$$

Using

$$b(y) - b(x) = (f(y) - f(x))a(x) + f(y)(a(y) - a(x))$$

and the triangle inequality, we find for $a, b \in W^{\varepsilon,p}(0, 2\pi)$

$$\Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|a(x)|^p\,|f(y)-f(x)|^p}{|y-x|^{1+p\varepsilon}} dy\,dx\Big)^{\frac{1}{p}} \le |b|_{\varepsilon,p} + \|f\|_{L^\infty(\mathbb{T})}|a|_{\varepsilon,p} < \infty\,.$$

Because of (2.5) from Lemma 2.1, this implies $a(x) = 0$ for almost all $x \in \mathbb{T}$ and then $b = af = 0$.

□

3. 2D DOMAIN WITH LIMITED REGULARITY

Let $F(x) = 1 + \int_0^x f(t)dt$. Then $F \in C^1(\mathbb{T})$, $F' = f$, and $\frac{1}{2} < F(x) < \frac{3}{2}$.

The latter estimate follows easily from

$$|F(x) - 1| = |\sum_{k=1}^{\infty} a_k \tfrac{1-\cos(b_k x)}{b_k}| \le 2^{-q} \sum_{k=1}^{\infty} 2\,q^{-k} = 2^{1-q}\tfrac{1}{q-1} \le \frac{1}{2}\,.$$

We define now the C^1 domain $\omega \subset \mathbb{R}^2$ using polar coordinates (r, θ)

$$\omega = \{(r,\theta) \mid r < F(\theta)\}\,.$$

Proposition 3.1. *Let $p \ge 1$, $\varepsilon > 0$ and $u \in W^{\frac{1}{p}+\varepsilon,p}(\omega;\mathbb{C}^2)$ be such that its normal trace $n \cdot u$ vanishes on $\partial\omega$. Then $u = 0$ on $\partial\omega$. The same conclusion is valid when the tangential trace $n \times u$ vanishes on $\partial\omega$.*

Proof. (Following Filonov [7, §5]) The unit normal n on $\partial\omega$ has the Cartesian components

$$n_1 = (F^2 + f^2)^{-\frac{1}{2}}(F\cos\theta + f\sin\theta), \quad n_2 = (F^2 + f^2)^{-\frac{1}{2}}(F\sin\theta - f\cos\theta)\,.$$

Therefore the condition $n_1u_1 + n_2u_2 = 0$ implies $af = b$ if we define

$$a = u_2\cos\theta - u_1\sin\theta\,, \quad b = (u_1\cos\theta + u_2\sin\theta)F$$

Now, since the traces u_j on $\partial\omega$, understood as functions $\theta \mapsto u_j(F(\theta), \theta)$ on $\mathbb{T}$, belong to $W^{\varepsilon,p}(\mathbb{T})$, we also have $a, b \in W^{\varepsilon,p}(\mathbb{T})$. According to Proposition 2.2 we find $a = b = 0$, which implies $u_1 = u_2 = 0$ on $\partial\omega$. The result using vanishing tangential trace follows by a rotation by $\pi/2$. □

Corollary 3.2. *(i) There exists $g \in C^\infty(\overline{\omega})$ such that the solution $v_D \in H^1_0(\omega)$ of the Dirichlet problem*

$$\Delta v_D = g \;\; in\, \omega\,; \quad v_D = 0 \;\; on\, \partial\omega$$

does not belong to $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$ for any $\varepsilon > 0$, $p \ge 1$.
(i) There exists $g \in C^\infty(\overline{\omega})$ such that any solution $v_N \in H^1(\omega)$ of the Neumann problem

$$\Delta v_N = g \;\; in\, \omega\,; \quad \partial_n v_N = 0 \;\; on\, \partial\omega$$

does not belong to $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$ for any $\varepsilon > 0$, $p \ge 1$.

Proof. For v_D one can take $g = 1$. Set $u = \nabla v_D$. If $v_D \in W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$, then u satisfies the hypotheses of Proposition 3.1 with vanishing tangential trace. Hence also the normal trace of u vanishes, i.e. $\partial_n v_D = 0$ on $\partial\omega$. Then Green's formula implies $\int_\omega g = 0$, which is not the case.

For $v_N \in W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$ one obtains similarly that the tangential derivative on the boundary vanishes, hence the trace of v_N on $\partial\omega$ is constant, without loss of generality equal to zero. Thus v_N is also solution of the Dirichlet problem. That there exists $g \in L^2(\omega)$ for which this is impossible can be seen as follows:

Let g be a non-zero harmonic polynomial such that $\int_\omega g = 0$, for example $g(x_1, x_2) = \alpha x_1 x_2 + \beta(x_1^2 - x_2^2)$ with suitably chosen coefficients $\alpha, \beta \in \mathbb{R}$. Then v_N exists, and Green's formula gives the contradiction

$$0 = \int_{\partial\omega} (\partial_n v_N\, g - v_N \partial_n g) ds = \int_\omega (\Delta v_N\, g - v_N \Delta g) dx = \int_\omega g^2 dx\,.$$

□

Remark 3.3. No eigenfunction of the Laplacian with Dirichlet conditions on ω can belong to $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$ with $\varepsilon > 0$, because it would also have vanishing normal derivative. Its extension by zero outside ω would then be a Dirichlet eigenfunction with the same eigenvalue on any domain containing ω. This contradicts for example the well known behavior of Dirichlet eigenvalues on disks or squares with varying size. It contradicts also the well known interior analyticity of Dirichlet eigenfunctions.

4. Example in higher dimensions

From $\omega \subset \mathbb{R}^2$ one can construct $\Omega \subset \mathbb{R}^d$ as follows (see [7], for $n = 3$ also [4, §6]). In cylindrical coordinates (r, θ, z), $z \in \mathbb{R}^{d-2}$:

$$\Omega = \{(r, \theta, z) \mid \frac{r^2}{F(\theta)^2} + |z|^2 < 1\}$$

The intersection with the plane $z = z_0$ gives for $|z_0| < 1$ the scaled domain $\sqrt{1 - |z_0|^2}\,\omega$. One can still prove that for this domain Ω and $0 < \varepsilon < 1$ there holds

$$W^{1+\frac{1}{p}+\varepsilon,p}(\Omega) \cap W_0^{1,p}(\Omega) = W_0^{1+\frac{1}{p}+\varepsilon,p}(\Omega)\,, \tag{4.1}$$

that is, for functions in $W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ the vanishing of the boundary trace implies that also the normal derivative is zero on the boundary.

Indeed, suppose that $v \in W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$, $v = 0$ on $\partial\Omega$ and let $u = \nabla v$. Then the tangential components of u are zero on the boundary, and we have to show that the normal component of u vanishes, too, on $\partial\Omega$. Define

$$\tilde{u}(r, \theta, z) = u(\sqrt{1 - |z|^2}\, r, \theta, z)\,.$$

Then $\tilde{u}$ is defined on the product domain

$$\tilde{\Omega} = \omega \times B_1 = \{(r, \theta, z) \mid (r, \theta) \in \omega, |z| < 1\}\,.$$

For any $\delta \in (0, 1)$, let $\tilde{\Omega}_\delta = \omega \times B_\delta$. On this product domain, there holds the inclusion

$$W^{s,p}(\tilde{\Omega}_\delta) \subset L^p\big(B_\delta; W^{s,p}(\omega)\big)\,,$$

as can be seen first for integer s directly from the definition of the Sobolev space $W^{s,p}$ and then for all $s \geq 0$ by interpolation. Thus $u \in W^{\frac{1}{p}+\varepsilon,p}(\Omega; \mathbb{C}^d)$ implies that $\tilde{u}$ restricted to $\tilde{\Omega}_\delta$ belongs to

$L^p\big(B_\delta; W^{\frac{1}{p}+\varepsilon,p}(\omega;\mathbb{C}^d)\big)$, and for almost every $z_0 \in B_\delta$, the restriction w_{z_0} of $\tilde u$ to the plane $z = z_0$ belongs to $W^{\frac{1}{p}+\varepsilon,p}(\omega,\mathbb{C}^d)$. The vanishing of the tangential components of u on $\partial\Omega$ implies that the component of w_{z_0} that is parallel to the plane $z = 0$ and tangential to $\partial\omega$ vanishes on $\partial\omega$. Then Proposition 3.1 tells us that the component of w_{z_0} that is parallel to the plane $z = 0$ and normal to $\partial\omega$ vanishes on $\partial\omega$, too. This means that at such a point $(r,\theta,z) \in \partial\Omega$ with $(\sqrt{1-|z|^2}\,r,\theta) \in \partial\omega$, $z = z_0$, in addition to the tangential components a component of u vanishes that is not tangential, and hence all components of u vanish there. Since this is true for almost all z_0 satisfying $|z_0| < \delta$ and for all $0 < \delta < 1$, we see that the trace of u on $\partial\Omega$ is zero, which proves (4.1).

The non-regularity result of Theorem 1.2 for the Dirichlet problem in Ω then follows in the same way as in the two-dimensional case. In particular, one can take $g = 1$ for the counterexample.

For the Neumann problem, a slightly different variant of adding $d-2$ variables works, and this variant could also be used for the Dirichlet problem, giving a counterexample with a somewhat less regular right hand side g. For this variant, (4.1) still holds. We redefine the domain Ω so that it contains a cylindrical part (see also [7, §5.2]). This is done by modifying the function $1-|z|^2$ in the previous example. Choose a decreasing C^∞ function μ on $\mathbb{R}_+$ satisfying

$$\mu(t) = 1 \ \text{ for } t \le 1\,; \qquad \mu(t) \le 0 \ \text{ for } t \ge 4\,; \qquad \mu'(t) < 0 \ \text{ for } t \ge 2\,,$$

and define

$$\Omega = \{(r,\theta,z) \mid r^2 < \mu(|z|^2)\,F(\theta)^2\}\,. \tag{4.2}$$

It is not hard to see that Ω has a C^1 boundary.

We now use the two-dimensional example presented in the previous section and denote by v_0 the function found there that satisfies the Neumann problem on ω with right hand side $g_0 \in C^\infty(\overline{\omega})$ and that does not belong to any $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$ for $\varepsilon > 0$, $p \ge 1$. In addition, we choose a function $\chi \in C_0^\infty(\overline{\mathbb{R}_+})$ satisfying $\chi(t) = 1$ for $t < \frac{1}{2}$, $\chi(t) = 0$ for $t \ge 1$. Then we define

$$v(x,z) = v_0(x)\,\chi(|z|); \qquad g(x,z) = g_0(x)\,\chi(|z|) + v_0(x)\Delta_z\chi(|z|); \qquad (x \in \omega,\ |z| < 1)\,.$$

Initially, v and g are defined on the cylinder $\omega \times B_1 \subset \Omega$, and we extend them by zero on the rest of Ω.

One easily verifies that v satisfies

$$\Delta v = g \ \text{ in } \Omega\,; \qquad \partial_n v = 0 \ \text{ on } \partial\Omega\,.$$

Noting that both $\chi(|z|)$ and $\Delta_z\chi(|z|)$ define $C^\infty(\overline{\Omega})$ functions and using the regularity of $v_0 \in W^{1+\frac{1}{p},p}(\omega)$ for all $p > 1$, so that v_0 is Hölder continuous on $\overline{\omega}$, one finds that g is Hölder continuous on $\overline{\Omega}$. Finally the non-regularity of v_0 implies clearly that also $v \notin W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ for $\varepsilon > 0$, $p \ge 1$.

This concludes the proof of Theorem 1.2.

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