Volume and surface integral equations for electromagnetic scattering by a dielectric body

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Abstract

We derive and analyze two equivalent integral formulations for the time-harmonic electromagnetic scattering by a dielectric object. One is a volume integral equation (VIE) with a strongly singular kernel and the other one is a coupled surface-volume system of integral equations with weakly singular kernels. The analysis of the coupled system is based on standard Fredholm integral equations, and it is used to derive properties of the volume integral equation.

Key words: Electromagnetic scattering, volume integral equation, dielectric interface problem

1. Introduction

We consider the solution via the integral equation method of the problem of electromagnetic scattering by a dielectric body. The scientific literature is abundant on the theoretical and numerical analysis of *surface* integral equations related to scattering problems. Conversely, the *volume* integral equation (VIE) using the strongly singular fundamental solution of Maxwell's equations has been the subject of only a few studies; see for example [1], [2] and [8], where the VIE is numerically solved with the method of moments. In [4], the VIE is combined with a multilevel fast multipole algorithm, to analyze antenna radiation in the presence of dielectric radomes. The spectrum of the volume integral operator is numerically studied in [3] and [12]. In [3] a spectral analysis is given under the hypothesis of Hölder continuity of constitutive parameters in the whole space. Likewise, the Lippmann-Schwinger equation studied in [5], which corresponds in the Maxwell case to our VIE, is analyzed there for a scattering problem in a medium with a refractive index uniformly Hölder continuously differentiable in the whole space $\mathbb{R}^3$.

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The assumption of global continuity is not realistic in the situation of the scattering by a dielectric, where the permittivity typically is discontinuous on the surface of the scatterer. Of practical importance are also composite dielectric materials with several surfaces of discontinuity. On the other hand, the magnetic permeability is often constant in this situation. Our contribution is the rigorous mathematical derivation of the VIE under the realistic hypothesis of discontinuity of the electric permittivity across the dielectric boundary. Moreover, we establish mapping properties and well-posedness of the VIE in standard function spaces associated with the electromagnetic energy, and we give first results about the essential spectrum of the volume integral operator in the space $L^2(\Omega)$, and in particular a Gårding inequality which is of importance for the stability of numerical algorithms based on the Galerkin method.

The VIE is also introduced in [1], for the scattering by a dielectric with discontinuities in the electric permittivity and the magnetic permeability of the medium. Such a volume integral equation is also used in [7] for the analysis of the far-field operator in dielectric scattering. In that paper, the integral equation is studied in $H(\mathrm{curl}, \Omega)$, and conditions for the material coefficients are given under which existence and uniqueness can be shown.

Our analysis of the VIE uses the equivalence with a coupled surface-volume system of integral equations which has only weakly singular kernels and is therefore easier to analyze. When the permittivity is continuous across the boundary, the boundary part of this coupled system disappears and one is left with the weakly singular form of the Lippmann-Schwinger equation that has already been investigated in [5]. The original scattering problem is equivalent to both integral formulations, and all three problems are well posed under realistic assumptions on the coefficients. While it is easy to see that the strongly singular volume integral operator has a non-trivial essential spectrum, a more complete study of its spectral properties is still to be done.

The needed technical tools are all available in standard references, such as the Stratton-Chu integral representation theorem in [5], the basic properties of the Sobolev spaces associated with the electromagnetic energy in [6], trace theorems and mapping properties of singular integral operators between Sobolev spaces in [10]. We also use the unique continuation principle from [9] or [11].

2. The problem

Let Ω^- be a bounded domain in $\mathbb{R}^3$ representing the dielectric scatterer. We use the notation $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega^-}$ and $\Gamma = \partial\Omega^-$, and we assume that the boundary Γ is regular (at least C^2). $\boldsymbol{n}$ is the unit outward normal vector to Ω^-.

The electric permittivity ε is a function of the space variable satisfying $\varepsilon(x) > 0$, $x \in \mathbb{R}^3$; $\varepsilon_{|_{\Omega^-}} \in C^1(\Omega^-) \cap C^0(\overline{\Omega^-})$; $\varepsilon_{|_{\Omega^+}} = \varepsilon_0$; and ε is discontinuous across Γ, in general. The vacuum permittivity ε_0 is a positive constant. We will denote the relative permittivity by $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0}$. We will also use the notation $\eta = 1 - \varepsilon_r$. The electric conductivity σ vanishes everywhere. We assume for simplicity that the magnetic permeability μ is constant ($\mu \equiv \mu_0 > 0$). With the frequency ω, the wave number is $\kappa = \omega\sqrt{\varepsilon_0 \mu_0} > 0$.

We use the function spaces:

$$
\begin{aligned}
&H(\mathrm{curl},\Omega^-) = \{\boldsymbol{u} \in L^2(\Omega^-)^3; \nabla\times\boldsymbol{u} \in L^2(\Omega^-)^3\},\\
&H(\mathrm{curl},\mathrm{div},\Omega^-) = H(\mathrm{curl},\Omega^-) \cap H(\mathrm{div},\Omega^-),\\
&H_{loc}(\mathrm{curl},(\overline{\Omega^+})) = \{\boldsymbol{u} \in L^2_{loc}(\overline{\Omega^+})^3; \nabla\times\boldsymbol{u} \in L^2_{loc}(\overline{\Omega^+})^3\},\\
&H_{loc}(\mathrm{curl},\mathrm{div},(\overline{\Omega^+})) = H_{loc}(\mathrm{curl},(\overline{\Omega^+})) \cap H_{loc}(\mathrm{div},(\overline{\Omega^+})).
\end{aligned}
$$

$H(\mathrm{div},\Omega^-)$ and $H(\mathrm{div},\Omega^+)$ (respectively $H_{loc}(\mathrm{div},(\overline{\Omega^+}))$) are defined in the same way as $H(\mathrm{curl},\Omega^-)$ (respectively $H_{loc}(\mathrm{curl},(\overline{\Omega^+}))$), with $\nabla\times\boldsymbol{u}$ replaced by $\nabla\cdot\boldsymbol{u}$.
As abbreviations for the restrictions onto the boundary Γ we write for the trace and the normal derivative of a scalar function u

$$
\gamma_0 u = u_{|\Gamma} \quad \text{and} \quad \gamma_1 u = \boldsymbol{n}\cdot\nabla u_{|\Gamma}\,,
$$

and for the normal and tangential traces of a vector function $\boldsymbol{u}$

$$
\gamma_n \boldsymbol{u} = \boldsymbol{n}\cdot\boldsymbol{u}_{|\Gamma} \quad \text{and} \quad \gamma_\times \boldsymbol{u} = \boldsymbol{n}\times\boldsymbol{u}_{|\Gamma}\,.
$$

Let $\boldsymbol{F} \in H(\mathrm{div},\,\Omega^+)$ be a vector field with a compact support contained in Ω^+, representing a current density that serves as source for the incident field scattered by the dielectric body Ω^-.

The scattering problem $(\mathcal{P})$ we want to solve can be written as follows:

Find $\boldsymbol{E}$, $\boldsymbol{H}$ such that $\boldsymbol{E}_i \in H(\mathrm{curl},\mathrm{div},\Omega^-), \boldsymbol{E}_e \in H_{loc}(\mathrm{curl},\mathrm{div},\overline{\Omega^+})$, $\boldsymbol{H}_i \in H(\mathrm{curl},\Omega^-), \boldsymbol{H}_e \in H_{loc}(\mathrm{curl},\overline{\Omega^+})$, with $\boldsymbol{E}_i = \boldsymbol{E}_{|_{\Omega^-}}, \boldsymbol{H}_i = \boldsymbol{H}_{|_{\Omega^-}}$, $\boldsymbol{E}_e = \boldsymbol{E}_{|_{\Omega^+}}$ and $\boldsymbol{H}_e = \boldsymbol{H}_{|_{\Omega^+}}$, satisfying the equations

$$
(\mathcal{P}) \quad
\begin{cases}
\nabla\times\boldsymbol{E}_i - i\kappa\,\boldsymbol{H}_i = 0 \quad \text{and} \quad \nabla\times\boldsymbol{H}_i + i\kappa\varepsilon_r\,\boldsymbol{E}_i = 0 & \text{in } \Omega^-,\\
\nabla\times\boldsymbol{E}_e - i\kappa\,\boldsymbol{H}_e = 0 \quad \text{and} \quad \nabla\times\boldsymbol{H}_e + i\kappa\,\boldsymbol{E}_e = \boldsymbol{F} & \text{in } \Omega^+,\\
\boldsymbol{n}\times\boldsymbol{H}_e = \boldsymbol{n}\times\boldsymbol{H}_i \quad \text{and} \quad \boldsymbol{n}\cdot\boldsymbol{H}_e = \boldsymbol{n}\cdot\boldsymbol{H}_i & \text{on } \Gamma,\\
\boldsymbol{n}\times\boldsymbol{E}_e = \boldsymbol{n}\times\boldsymbol{E}_i \quad \text{and} \quad \boldsymbol{n}\cdot\boldsymbol{E}_e = \boldsymbol{n}\cdot\varepsilon_r\,\boldsymbol{E}_i & \text{on } \Gamma,\\
\boldsymbol{H}_e\times\dfrac{\boldsymbol{x}}{r} - \boldsymbol{E}_e = \mathcal{O}\left(\dfrac{1}{r^2}\right),\ r = |\boldsymbol{x}| \to +\infty.
\end{cases}
$$

Note that the interface conditions simply express the fact that $\nabla\times\boldsymbol{E}$, $\nabla\times\boldsymbol{H}$, $\nabla\cdot\boldsymbol{H}$ and $\nabla\cdot(\varepsilon\boldsymbol{E})$ are locally integrable, that is that the time-harmonic Maxwell equations are satisfied in the distributional sense in the whole space. The interface conditions on the normal components are a consequence of the conditions on the tangential components and of the Maxwell equations in $\Omega^-\cup\Omega^+$, and therefore the interface problem $(\mathcal{P})$ is often equivalently formulated without the interface conditions on the normal components.

The physical situation described by problem $(\mathcal{P})$ is the electromagnetic field radiated by an antenna and refracted by a dielectric lens. A slightly different scattering problem is often considered in the literature where the incident field is given, for example as a plane wave, and only the scattered field is considered in Ω^+. This leads to a mathematically equivalent formulation where the differential equations are homogeneous ($\boldsymbol{F} = 0$) and the transmission conditions are inhomogeneous.

3. Integral formulations

As a first step in the derivation of the integral equations, we extend the well-known Stratton-Chu integral representation to fields $(\boldsymbol{E}, \boldsymbol{H})$ in $H(\mathrm{curl}, \mathrm{div}, \Omega^-) \times H(\mathrm{curl}, \Omega^-)$.

Lemma 1 (Stratton-Chu) *Let D be a C^2 regular bounded domain in $\mathbb{R}^3$, $\boldsymbol{n}$ the unit outward normal to ∂D, $\boldsymbol{E}$ and $\boldsymbol{H}$ two vector fields in $C^1(\overline{D})$. Then for all $x \in D$ there holds*

$$
\begin{aligned}
\boldsymbol{E}(x) = -\nabla \times \int_{\partial D} \boldsymbol{n}(y) \times \boldsymbol{E}(y) G_\kappa(x-y) ds(y) + \nabla \int_{\partial D} \boldsymbol{n}(y) \cdot \boldsymbol{E}(y) G_\kappa(x-y) ds(y) \\
-i\kappa \int_{\partial D} \boldsymbol{n}(y) \times \boldsymbol{H}(y) G_\kappa(x-y) ds(y) + i\kappa \int_D \{\nabla \times \boldsymbol{H}(y) + i\kappa\, \boldsymbol{E}(y)\} G_\kappa(x-y) dy \\
-\nabla \int_D \nabla \cdot \boldsymbol{E}(y) G_\kappa(x-y) dy + \nabla \times \int_D \{\nabla \times \boldsymbol{E}(y) - i\kappa\, \boldsymbol{H}(y)\}\, G_\kappa(x-y) dy
\end{aligned} \tag{1}
$$

with $G_\kappa(x-y) = \dfrac{e^{i\kappa|x-y|}}{4\pi|x-y|}$, the fundamental solution of the Helmholtz equation.
This representation holds also for $(\boldsymbol{E}, \boldsymbol{H})$ in $H(\mathrm{curl}, \mathrm{div}, D) \times H(\mathrm{curl}, D)$, where the boundary values are understood in the sense of weak tangential and normal traces in $H^{-1/2}(\Gamma)$.

For a proof of the regular case, see [5], page 156.

In order to see that the formula (1) is also valid for $(\boldsymbol{E}, \boldsymbol{H}) \in H(\mathrm{curl}, \mathrm{div}, D) \times H(\mathrm{curl}, D)$, we use the density of smooth functions in these spaces [6] and the continuity of the integral operators:

Let us introduce the integral operators of the volume potential $\mathcal{N}$ and the single layer potential $\mathcal{S}$, both acting on scalar functions as well as on vector fields:

$$
\mathcal{N}u(x) = \int_D u(y) G_\kappa(x-y)\, dy\ ; \quad \mathcal{S}f(x) = \int_{\partial D} f(y) G_\kappa(x-y)\, ds(y).
$$

With the normal and tangential traces on ∂D, $\gamma_n \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{u}_{|\partial D}$ and $\gamma_\times \boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{u}_{|\partial D}$. we can then write the relation (1) in the form

$$
\mathcal{K}(\boldsymbol{E}, \boldsymbol{H}) = 0 \qquad \forall\ \boldsymbol{E}, \boldsymbol{H} \in \left(C^1(\overline{D})\right)^3 \tag{2}
$$

where we set

$$
\begin{aligned}
\mathcal{K}(\boldsymbol{E}, \boldsymbol{H}) = \boldsymbol{E} + \nabla \times (\mathcal{S}\gamma_\times \boldsymbol{E}) - \nabla \mathcal{S}\gamma_n \boldsymbol{E} + i\kappa \mathcal{S}\gamma_\times \boldsymbol{H} \\
- \nabla \times \mathcal{N}(\nabla \times \boldsymbol{E} - i\kappa \boldsymbol{H}) + \nabla \mathcal{N}(\nabla \cdot \boldsymbol{E}) - i\kappa \mathcal{N}(\nabla \times \boldsymbol{H} + i\kappa \boldsymbol{E}).
\end{aligned}
$$

Since the operators

$$
\mathcal{S} : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^1(D) \quad \text{and } \mathcal{N} : L^2(D) \longrightarrow H^2(D)
$$

are continuous, we can extend (2) by density from $(C^1(\overline{D}))^6$ and obtain

$$
\mathcal{K}(\boldsymbol{E}, \boldsymbol{H}) = 0 \qquad \forall\ (\boldsymbol{E}, \boldsymbol{H}) \in H(\text{curl, div, D}) \times H(\text{curl, D})\ . \qquad \blacksquare
$$

Using this extended Stratton-Chu formula for $D = \Omega^-$ and for $D = \Omega^+ \cap B_R$, where the radius R of the ball B_R tends to infinity, together with the Maxwell equations of the problem $(\mathcal{P})$ and the radiation condition, we establish the following lemma:

Lemma 2 *Let $\boldsymbol{E}$ and $\boldsymbol{H}$ be two vector fields on $\mathbb{R}^3$ satisfying the hypotheses and equations of problem $(\mathcal{P})$ except for the interface conditions, and for $x \in \mathbb{R}^3$, let*

$$\begin{aligned}\boldsymbol{U}_i(x) = &-\nabla \times \mathcal{S}(\gamma_\times \boldsymbol{E}_i)(x) + \nabla \mathcal{S}(\gamma_n \boldsymbol{E}_i)(x) - i\kappa \mathcal{S}(\gamma_\times \boldsymbol{H}_i)(x)\\ &-\nabla \mathcal{N}(\nabla \cdot \boldsymbol{E}_i)(x) - \kappa^2 \mathcal{N}(\eta \boldsymbol{E}_i)(x)\end{aligned}$$

and

$$\boldsymbol{U}_e(x) = \nabla \times \mathcal{S}(\gamma_\times \boldsymbol{E}_e)(x) - \nabla \mathcal{S}(\gamma_n \boldsymbol{E}_e)(x) + i\kappa \mathcal{S}(\gamma_\times \boldsymbol{H}_e)(x) + \boldsymbol{D}(x)\,,$$

where

$$\boldsymbol{D}(x) = -\frac{1}{i\kappa}\nabla \int_{\Omega^+} \nabla\cdot \boldsymbol{F}(y) G_\kappa(x-y)\,dy + i\kappa \int_{\Omega^+} G_\kappa(x-y)\boldsymbol{F}(y)\,dy\,.$$

Then we have

$$\boldsymbol{U}_i = \begin{cases}\boldsymbol{E}_i & \text{in } \Omega^-\\ 0 & \text{in } \Omega^+\end{cases} \qquad \text{and} \qquad \boldsymbol{U}_e = \begin{cases}0 & \text{in } \Omega^-\\ \boldsymbol{E}_e & \text{in } \Omega^+\,.\end{cases}$$

Proposition 3 *Let $(\boldsymbol{E}, \boldsymbol{H})$ be solution of Problem $(\mathcal{P})$. Then we have the following two integral representations for $\boldsymbol{E}$ in $\mathbb{R}^3$:*

$$\boldsymbol{E} = \nabla \mathcal{S}(\eta\,\gamma_n \boldsymbol{E}_i) - \nabla \mathcal{N}(\nabla \cdot \boldsymbol{E}_i) - \kappa^2 \mathcal{N}(\eta \boldsymbol{E}_i) + \boldsymbol{D} \tag{3}$$

and

$$\boldsymbol{E} = \nabla \mathcal{M}(\eta \boldsymbol{E}_i) - \kappa^2 \mathcal{N}(\eta \boldsymbol{E}_i) + \boldsymbol{D} \tag{4}$$

Here, as in the previous lemma, the Newton potential $\mathcal{N}$ is defined with respect to integration over the interior domain Ω^-, and the operator $\mathcal{M}$ is given by

$$\mathcal{M}\boldsymbol{u}(x) = \int_{\Omega^-} \nabla_y\, G_\kappa(x-y) \cdot \boldsymbol{u}(y)\,dy\ .$$

Proof: From the previous lemma, we have in all of $\mathbb{R}^3$:

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{U}_i + \boldsymbol{U}_e\\ &= \nabla \times \mathcal{S}(\gamma_\times(\boldsymbol{E}_e - \boldsymbol{E}_i)) - \nabla \mathcal{S}(\gamma_n(\boldsymbol{E}_e - \boldsymbol{E}_i)) + i\kappa \mathcal{S}(\gamma_\times(\boldsymbol{H}_e - \boldsymbol{H}_i))\\ &\quad -\nabla \mathcal{N}(\nabla \cdot \boldsymbol{E}_i) - \kappa^2 \mathcal{N}(\eta \boldsymbol{E}_i) + \boldsymbol{D}\end{aligned}$$

Taking account of the boundary conditions:

$$\gamma_\times(\boldsymbol{E}_e - \boldsymbol{E}_i) = 0 = \gamma_\times(\boldsymbol{H}_e - \boldsymbol{H}_i) \text{ and } \gamma_n(\boldsymbol{E}_e - \boldsymbol{E}_i) = (\varepsilon_r - 1)\gamma_n \boldsymbol{E}_i \text{ on } \Gamma,$$

we arrive at the first integral representation (3).
Furthermore, an integration by parts gives

$$\begin{aligned}\mathcal{S}(\eta\,\boldsymbol{n}\cdot\boldsymbol{E}_i)(x) &= \int_\Gamma (1-\varepsilon_r(y)) G_\kappa(x-y)\,\boldsymbol{n}(y)\cdot \boldsymbol{E}_i(y)\,ds(y)\\ &= \int_{\Omega^-} G_\kappa(x-y)\,\nabla\cdot \boldsymbol{E}_i(y)\,dy - \int_{\Omega^-} G_\kappa(x-y)\,\nabla\cdot(\varepsilon_r(y)\boldsymbol{E}_i(y))\,dy\\ &\qquad + \int_{\Omega^-} (1-\varepsilon_r(y))\nabla_y G_\kappa(x-y)\cdot \boldsymbol{E}_i(y)\,dy.\end{aligned}$$

Hence, using $\nabla\cdot(\varepsilon_r \boldsymbol{E}_i) = 0$ we obtain

$$\mathcal{S}(\eta\,\gamma_n \boldsymbol{E}_i) - \mathcal{N}(\nabla\cdot\boldsymbol{E}_i) = \mathcal{M}(\eta \boldsymbol{E}_i)\ . \tag{5}$$

Injecting this equality into (3), we get the representation (4). ■

From this proposition we will derive two integral equations, a coupled surface-volume system and a volume equation.

Recall first that from the equations of the problem $(\mathcal{P})$, we get $\nabla\cdot(\varepsilon_r \boldsymbol{E}_i) = 0$ in the sense of distributions on $\mathbb{R}^3$, hence in Ω^- there holds $\nabla\cdot\boldsymbol{E}_i = -\dfrac{1}{\varepsilon_r}\,\nabla\varepsilon_r\cdot\boldsymbol{E}_i$, and therefore $\nabla\cdot\boldsymbol{E}_i$ can be replaced by $-\dfrac{1}{\varepsilon_r}\,\nabla\,\varepsilon_r\cdot\boldsymbol{E}_i$ in the above integral representation (3). Let us denote by $\boldsymbol{\tau}$ this logarithmic gradient of ε_r : $\boldsymbol{\tau} = -\dfrac{1}{\varepsilon_r}\,\nabla\varepsilon_r$.

The following integral operators appear in addition to the operator $\mathcal{M}_\eta : \ \boldsymbol{u}\mapsto\mathcal{M}(\eta\boldsymbol{u})$:

$$\mathcal{S}_\eta \ : \ f\mapsto\mathcal{S}(\eta\, f);\quad \mathcal{N}_{\boldsymbol{\tau}} \ : \ \boldsymbol{u}\mapsto\mathcal{N}(\boldsymbol{\tau}\cdot\boldsymbol{u});\quad \mathcal{N}_\eta \ : \ \boldsymbol{u}\mapsto\mathcal{N}(\eta\boldsymbol{u})\ . \tag{6}$$

where f and $\boldsymbol{u}$ are respectively scalar and vector fields defined on Γ and on Ω^-.

We also need the one-sided traces

$$\gamma_0^\pm g := g^\pm_{|\Gamma},\quad \gamma_1^\pm g := (\boldsymbol{n}\cdot\nabla g^\pm)_{|\Gamma} \ \text{ and } \ \gamma_n^\pm \boldsymbol{v} := \gamma_n \boldsymbol{v}^\pm,$$

for g and $\boldsymbol{v}$ respectively scalar and vector fields defined on $\mathbb{R}^3$, with $g^\pm := g_{|\Omega^\pm}$ and $\boldsymbol{v}^\pm := \boldsymbol{v}_{|\Omega^\pm}$.

The coupled surface-volume system of integral equations is given by the problem $(\boldsymbol{\mathcal{E}_1})$ defined as follows:

$$(\boldsymbol{\mathcal{E}_1})\qquad \begin{cases} \text{Find } (\boldsymbol{E}_*, e_*) \in (L^2(\Omega^-))^3 \times H^{-\frac{1}{2}}(\Gamma), \text{ such that} \\ \begin{pmatrix} 1 - \nabla\mathcal{N}_{\boldsymbol{\tau}} + \kappa^2\mathcal{N}_\eta & -\nabla\mathcal{S}_\eta \\ \kappa^2\gamma_n^-\mathcal{N}_\eta - \gamma_1^-\mathcal{N}_{\boldsymbol{\tau}} & 1 - \gamma_1^-\mathcal{S}_\eta \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_* \\ e_* \end{pmatrix} = \begin{pmatrix} \boldsymbol{D} \\ \gamma_n^-\boldsymbol{D} \end{pmatrix} \end{cases}$$

and the VIE is given by the problem $(\boldsymbol{\mathcal{E}_2})$ defined as follows:

$$(\boldsymbol{\mathcal{E}_2})\qquad \begin{cases} \text{Find } \boldsymbol{E}_\circ \in (L^2(\Omega^-))^3, \text{ such that} \\ \left(1 - \nabla\mathcal{M}_\eta + \kappa^2\mathcal{N}_\eta\right)\boldsymbol{E}_\circ = \boldsymbol{D}\ . \end{cases}$$

Remark 4 A quicker, if less rigorous, way of arriving at the second integral representation (4) and from there by restriction to Ω^- at the VIE $(\mathcal{E}_2)$, is the following:

Write the Maxwell transmission problem $(\mathcal{P})$ as a second order system, valid in the distributional sense on the whole space, and move the inhomogeneity to the right hand side:

$$\nabla\times(\nabla\times\boldsymbol{E}) - \kappa^2\boldsymbol{E} = -\kappa^2\eta\boldsymbol{E} + i\kappa\boldsymbol{F}\,.$$

Then solve this system by convolution with the (strongly singular) fundamental solution $\boldsymbol{U}^*$ of the constant-coefficient operator $\nabla \times (\nabla \times) - \kappa^2$:

$$\boldsymbol{U}^*(x) = \frac{1}{\kappa^2} \nabla \nabla G_\kappa(x) + G_\kappa(x) \,.$$

This gives

$$\boldsymbol{E} = -\kappa^2 \boldsymbol{U}^* * (\eta \boldsymbol{E}) + i\kappa \boldsymbol{U}^* * \boldsymbol{F}$$

which, by noticing that η vanishes outside of Ω^-, can be seen to coincide with the representation formula (4).

4. Equivalence results and well-posedness

We prove equivalence between the scattering problem and the integral formulations, via the following theorems.

Theorem 5 *If $(\boldsymbol{E}, \boldsymbol{H})$ is a solution of the problem $(\mathcal{P})$, then $(\boldsymbol{E}_i, \gamma_n \boldsymbol{E}_i)$ is a solution of the problem $(\mathcal{E}_1)$.*

Proof: This is a direct consequence of the previous proposition. Indeed, applying the formula (3) to the restriction $\boldsymbol{E}_i = \boldsymbol{E}_{|_{\Omega^-}}$, and remembering that $\nabla \cdot \boldsymbol{E}_i = -\boldsymbol{\tau} \cdot \boldsymbol{E}_i$, we get the first equation of the problem $(\mathcal{E}_1)$. The second one is obtained by taking the normal trace of the first equation on the boundary. So the couple $(\boldsymbol{E}_i, \gamma_n \boldsymbol{E}_i)$ is a solution of $(\mathcal{E}_1)$, because it belongs to $(L^2(\Omega^-))^3 \times H^{-\frac{1}{2}}(\Gamma)$. $\blacksquare$

Conversely, we have:

Theorem 6 *If $(\boldsymbol{E}_*, e_*) \in (L^2(\Omega^-))^3 \times H^{-\frac{1}{2}}(\Gamma)$ is a solution of the problem $(\mathcal{E}_1)$, then we have a solution $(\boldsymbol{E}, \boldsymbol{H})$ of the problem $(\mathcal{P})$ by defining:*

$$\boldsymbol{E}_{|_{\Omega^-}} = \boldsymbol{E}_*,$$

$$\boldsymbol{E}_{|_{\Omega^+}}(x) = \nabla \mathcal{S}(\eta\, e_*)(x) - \nabla \mathcal{N}(\nabla \cdot \boldsymbol{E}_*)(x) - \kappa^2 \mathcal{N}(\eta\, \boldsymbol{E}_*)(x) + \boldsymbol{D}(x) \,,$$

$$\boldsymbol{H}_{|_{\Omega^-}} = \frac{1}{i\kappa} \nabla \times \boldsymbol{E}_* \qquad \textit{and} \qquad \boldsymbol{H}_{|_{\Omega^+}} = \frac{1}{i\kappa} \nabla \times \boldsymbol{E}_{|_{\Omega^+}} .$$

Proof: From the definition of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ and the continuity properties of the corresponding integral operators it is clear that the fields belong to the function spaces required for solutions of the problem $(\mathcal{P})$. The Silver-Müller radiation condition is a consequence of the asymptotic behavior of the integral kernels at infinity.
We now check that the Maxwell equations are satisfied. The equations $\nabla \times \boldsymbol{E}_* - i\kappa \boldsymbol{H}_{|_{\Omega^-}} = 0$ and $\nabla \times \boldsymbol{E}_{|_{\Omega^+}} - i\kappa \boldsymbol{H}_{|_{\Omega^+}} = 0$ are satisfied by definition. Furthermore, we have $\nabla \times \boldsymbol{H}_{|_{\Omega^-}} + i\kappa\varepsilon_r \boldsymbol{E}_{|_{\Omega^-}} = \frac{1}{i\kappa} \nabla \times (\nabla \times \boldsymbol{E}_*) + i\kappa\varepsilon_r \boldsymbol{E}_*$. Using integration by parts, the relation $\nabla \times (\nabla \times) = -\Delta + \nabla(\nabla \cdot)$ and the equality $\displaystyle\int_{\Omega^+} \nabla_y \cdot (G_\kappa(x-y) \boldsymbol{F}(y))\, dy = 0$ which is valid because $\mathrm{Supp} \boldsymbol{F} \subset \Omega^+$, we get from $(\mathcal{E}_1)$

$$(\nabla \times \boldsymbol{H}_{|_{\Omega^-}} + i\kappa\varepsilon_r \boldsymbol{E}_{|_{\Omega^-}})(x) = i\kappa \nabla \int_{\Omega^-} (1 - \varepsilon_r(y)) G_\kappa(x-y) q(y) dy \tag{7}$$

with $q = \dfrac{1}{\varepsilon_r}\,\nabla\cdot(\varepsilon_r \boldsymbol{E}_*)$. We have to show that $q = 0$.
Taking the divergence in (7), we see that q is a solution u of the scalar Lippmann-Schwinger equation:

$$\begin{cases} \text{Find u} \in L^2(\Omega^-), \text{ such that} \\ \qquad\qquad (1+\kappa^2\mathcal{N}_\eta)u = 0 \text{ in } \quad \Omega^- \end{cases} \tag{8}$$

Lemma 7 *The trivial solution is the unique solution of the problem (8).*

Proof of the Lemma: From $u = -\kappa^2\mathcal{N}_\eta u$ we find $u \in H^2(\Omega^-)$, since $\mathcal{N}$ is bounded from $L^2(\Omega^-)$ to $H^2(\Omega^-)$. We can define an extension of u to all of $\mathbb{R}^3$ by $v(x) := -\kappa^2 \displaystyle\int_{\Omega^-} \eta(y))G_\kappa(x-y)u(y)\,dy$. Since $v \in H^2_{loc}(\mathbb{R}^3)$, we have

$$[\gamma_0 v]_\Gamma := \gamma_0^+(v) - \gamma_0^-(v) = 0, \text{ and } \ [\gamma_1 v]_\Gamma := \gamma_1^+(v) - \gamma_1^-(v) = 0.$$

Thus v is solution of the problem:

$$\begin{cases} v \in H^2_{loc}(\mathbb{R}^3) \\ (\Delta + \kappa^2\varepsilon_r)v \ = 0 \quad \text{in}\, \mathbb{R}^3\ ; \qquad \partial_r v - i\kappa v = \mathcal{O}\left(\dfrac{1}{r^2}\right),\ r \to +\infty \end{cases}$$

The Sommerfeld radiation condition and the Rellich lemma show that v vanishes outside Ω^-. Then using the unique continuation principle ([9], page 65) in a domain strictly containing Ω^-, we get that v vanishes everywhere. and in particular $u = 0$. This completes the proof of the Lemma. ∎

Coming back to (7) and using $q = 0$, we get $\nabla\times\boldsymbol{H}_{|\Omega^-} + i\kappa\varepsilon_r\,\boldsymbol{E}_{|\Omega^-} = 0$.
Moreover, we have

$$\nabla\times\boldsymbol{H}_{|\Omega^+} + i\kappa\,\boldsymbol{E}_{|\Omega^+} = \frac{1}{i\kappa}\,\nabla\times(\nabla\times\boldsymbol{E}_{|\Omega^+}) + i\kappa\,\boldsymbol{E}_{|\Omega^+}.$$

Noticing that $q = 0$ implies $\nabla\cdot\boldsymbol{E}_* = -\frac{1}{\varepsilon_r}\nabla\varepsilon_r\cdot\boldsymbol{E}_*$, integrating by parts in $\displaystyle\int_{\Omega^-} \nabla_y\cdot[(1-\varepsilon_r(y))G_\kappa(x-y)\boldsymbol{E}_*(y)]\;dy$ and using the relation $\frac{1}{i\kappa}\nabla\times(\nabla\times\boldsymbol{D}) + i\kappa\boldsymbol{D} = \boldsymbol{F}$, we get $\nabla\times\boldsymbol{H}_{|\Omega^+} + i\kappa\,\boldsymbol{E}_{|\Omega^+} = \boldsymbol{F}$. So the Maxwell equations are satisfied.
Let us now verify the interface conditions. Consider a ball $B_R = \{x \in \mathbb{R}^3; |x| < R\}$ with $R > 0$ such that $\overline{\Omega^-} \subset B_R$. We note $B_R^+ = B_R \setminus \overline{\Omega^-}$. For $\phi \in C_0^\infty(B_R)^3$ we have

$$\langle \gamma_\times\boldsymbol{E}_{|\Omega^+} - \gamma_\times\boldsymbol{E}_{|\Omega^-}, \phi\rangle_\Gamma = \int_{B_R^+} (\boldsymbol{E}_{|\Omega^+}\cdot\nabla\times\phi - \nabla\times\boldsymbol{E}_{|\Omega^+}\cdot\phi) + \int_{\Omega^-} (\boldsymbol{E}_*\cdot\nabla\times\phi - \nabla\times\boldsymbol{E}_*\cdot\phi).$$

Inserting the definition of $\boldsymbol{E}_{|\Omega^+}$ into this expression involves the following functions:

$$a = \mathcal{S}(\eta e_*), \quad b = \mathcal{N}(\nabla\cdot\boldsymbol{E}_*), \quad \boldsymbol{c} = \mathcal{N}(\eta\boldsymbol{E}_*),$$

$$\boldsymbol{d}(x) := \int_{\Omega^+} G_\kappa(x-y)\boldsymbol{F}(y)\,dy, \quad l(x) := \int_{\Omega^+} \nabla\cdot\boldsymbol{F}(y)G_\kappa(x-y)\,dy.$$

We have $a \in H^1(B_R)$. The functions b, l on one hand and $\boldsymbol{c}$, $\boldsymbol{d}$ on the other hand, are respectively in $H^2(B_R)$ and in $H^2(B_R)^3$, so their jumps at the boundary Γ vanish.

Therefore $\langle \gamma_\times \boldsymbol{E}_{|\Omega^+} - \gamma_\times \boldsymbol{E}_{|\Omega^-}, \phi\rangle_\Gamma = 0$, $\forall \phi \in C_0^\infty(B_R)^3$, hence $\gamma_\times \boldsymbol{E}_{|\Omega^+} = \gamma_\times \boldsymbol{E}_{|\Omega^-}$ in $H^{-\frac{1}{2}}(\Gamma)^3$. The functions a, b, $\boldsymbol{c}$, $\boldsymbol{d}$ and l appear also in the expressions of $\langle \gamma_\times \boldsymbol{H}_{|\Omega^+} - \gamma_\times \boldsymbol{H}_{|\Omega^-}, \phi\rangle_\Gamma$ and $\langle \gamma_n \boldsymbol{E}_{|\Omega^+} - \gamma_n \boldsymbol{E}_{|\Omega^-}, \psi\rangle_\Gamma$. With the same arguments we get $\langle \gamma_\times \boldsymbol{H}_{|\Omega^+} - \gamma_\times \boldsymbol{H}_{|\Omega^-}, \phi\rangle_\Gamma = 0$, $\forall \phi \in C_0^\infty(B_R)^3$ and $\langle \gamma_n \boldsymbol{H}_{|\Omega^+} - \gamma_n \boldsymbol{H}_{|\Omega^-}, \psi\rangle_\Gamma = 0$, $\forall \psi \in C_0^\infty(B_R)$, hence $\gamma_\times \boldsymbol{H}_{|\Omega^+} = \gamma_\times \boldsymbol{H}_{|\Omega^-}$ and $\gamma_n \boldsymbol{H}_{|\Omega^+} = \gamma_n \boldsymbol{H}_{|\Omega^-}$ in the sense of $H^{-\frac{1}{2}}(\Gamma)$. The functions $a, b, \boldsymbol{c}, \boldsymbol{d}$ and l are again involved in the expression of the normal components of the field $\boldsymbol{E}$. Since $b, l \in H^2(B_R)$ and $\boldsymbol{c}, \boldsymbol{d} \in H^2(B_R)^3$, we have $[\gamma_1 b]_\Gamma = [\gamma_1 l]_\Gamma = [\gamma_n \boldsymbol{c}]_\Gamma = [\gamma_n \boldsymbol{d}]_\Gamma = 0$. On the other hand, we find $[\gamma_1 a]_\Gamma = -\eta\, e_*$. Thus,

$$\langle \gamma_n \boldsymbol{E}_{|\Omega^+} - \gamma_n \varepsilon_r \boldsymbol{E}_{|\Omega^-}, \psi\rangle_\Gamma = \int_\Gamma \eta(x))\psi(x)\big[-e_*(x) + \gamma_1^- a(x) - \gamma_1^- b(x) \\ -\kappa^2 \gamma_n^- \boldsymbol{c}(x) + i\kappa\, \gamma_n^- \boldsymbol{d}(x) - \frac{1}{i\kappa}\gamma_1^- l(x)\big]\, ds(x).$$

From the expression of e_*, we have

$$-e_* + \gamma_1^- a - \gamma_1^- b - \kappa^2 \gamma_n^- \boldsymbol{c} + i\kappa\, \gamma_n^- \boldsymbol{d} - \frac{1}{i\kappa}\gamma_1^- l = 0.$$

So $\langle \gamma_n \boldsymbol{E}_{|\Omega^+} - \gamma_n \varepsilon_r \boldsymbol{E}_{|\Omega^-}, \psi\rangle_\Gamma = 0$, $\forall \psi \in C_0^\infty(B_R)$, hence $\gamma_n \boldsymbol{E}_{|\Omega^+} = \gamma_n \varepsilon_r \boldsymbol{E}_{|\Omega^-}$ in $H^{-\frac{1}{2}}(\Gamma)$). Thereby, the interface conditions are satisfied too. This completes the proof of the theorem. ∎

In Theorems 5 and 6 we showed equivalence between the scattering problem $(\mathcal{P})$ and the first integral formulation $(\mathcal{E}_1)$. In this context, the right hand side had a particular form coming from our assumption that the sources are situated in the exterior domain. Therefore the right hand side $\boldsymbol{D}$ in the integral equation was the field generated by such a source, and was therefore analytic on the whole domain $\overline{\Omega^-}$. In order to study mapping properties of the integral operators, in particular the strongly singular operator appearing in the VIE $(\mathcal{E}_2)$, we need to consider now more general right hand sides $\boldsymbol{D}$. The following equivalence theorem between the two integral formulations $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ holds in such a more general situation.

Theorem 8 *Let $\boldsymbol{D} \in H(\mathrm{div}, \Omega^-)$, $\nabla \cdot \boldsymbol{D} = 0$.*
(i) If $(\boldsymbol{E}_*, e_*) \in L^2(\Omega^-)^3 \times H^{-\frac{1}{2}}(\Gamma)$ is a solution of the problem $(\mathcal{E}_1)$, then $\boldsymbol{E}_*$ is a solution of the problem $(\mathcal{E}_2)$.
(ii) If $\boldsymbol{E}_\circ \in L^2(\Omega^-)^3$ is a solution of the probem $(\mathcal{E}_2)$, then $\boldsymbol{E}_\circ \in H(\mathrm{div}, \Omega^-)$ and defining $e_\circ = \gamma_n \boldsymbol{E}_\circ \in H^{-\frac{1}{2}}(\Gamma)$, the pair $(\boldsymbol{E}_\circ, e_\circ)$ is a solution of the probem $(\mathcal{E}_1)$.

Proof:
(i) Let $(\boldsymbol{E}_*, e_*)$ be a solution of the problem $(\mathcal{E}_1)$, then

$$\boldsymbol{E}_* = \nabla \mathcal{N}(\boldsymbol{\tau} \cdot \boldsymbol{E}_*) + \nabla \mathcal{S}(\eta e_*) - \kappa^2 \mathcal{N}(\eta \boldsymbol{E}_*) + \boldsymbol{D}.$$

It is easy to see that $\boldsymbol{E}_* \in H(\mathrm{div}, \Omega^-)$. From the second equation of the system $(\mathcal{E}_1)$, we see $e_* = \gamma_n \boldsymbol{E}_*$. As in the proof of Theorem 6, we conclude that $\nabla \cdot \boldsymbol{E}_* = -\boldsymbol{\tau} \cdot \boldsymbol{E}_*$, hence $\nabla \cdot (\varepsilon_r \boldsymbol{E}_*) = 0$, and we can integrate by parts as in (5) to get

$$\nabla \mathcal{N}(\boldsymbol{\tau} \cdot \boldsymbol{E}_*) + \nabla \mathcal{S}(\eta \gamma_n \boldsymbol{E}_*) = \nabla \mathcal{M}(\eta \boldsymbol{E}_*),$$

hence

$$\boldsymbol{E}_* = \nabla \mathcal{M}(\eta \boldsymbol{E}_*) - \kappa^2 \mathcal{N}(\eta \boldsymbol{E}_*) + \boldsymbol{D}.$$

Thus $\boldsymbol{E}_*$ is a solution of the problem $(\mathcal{E}_2)$.

(ii) Reciprocally, let $\boldsymbol{E}_\circ$ be a solution of the problem $(\mathcal{E}_2)$. We are first going to show that $\boldsymbol{E}_\circ \in H(\mathrm{div}, \Omega^-)$.
We write $\boldsymbol{E}_\circ = \mathcal{A}\boldsymbol{E}_\circ + \boldsymbol{D}$, with $\mathcal{A} = \nabla\mathcal{M}_\eta - \kappa^2\mathcal{N}_\eta$. Since $\mathcal{M}$ is bounded from $L^2(\Omega^-)^3$ to $H^1(\Omega^-)$ and $\mathcal{N}$ is bounded from $L^2(\Omega^-)^3$ to $H^2(\Omega^-)^3$, it is clear that $\mathcal{A}$ is bounded from $L^2(\Omega^-)^3$ to itself.

For $\boldsymbol{u} \in C_0^\infty(\Omega^-)^3$, a simple computation gives $\nabla\cdot\mathcal{A}\boldsymbol{u} = \nabla\cdot(\eta\boldsymbol{u})$; setting therefore $\mathcal{C}\boldsymbol{u} = \nabla\cdot(\mathcal{A}\boldsymbol{u} - \eta\boldsymbol{u})$, we have

$$\mathcal{C}\boldsymbol{u} = 0\,, \qquad \forall\, \boldsymbol{u} \in C_0^\infty(\Omega^-)^3\,. \tag{9}$$

The operator $\mathcal{C}$ is bounded from $L^2(\Omega^-)^3$ to $H^{-1}(\Omega^-)$. Thus, from the density of $C_0^\infty(\Omega^-)$ in $L^2(\Omega^-)$), we deduce that $\mathcal{C}\boldsymbol{u} = 0$ holds for all $\boldsymbol{u} \in L^2(\Omega^-)^3$. Therefore, we get for the solution $\boldsymbol{E}_\circ$ of $(\mathcal{E}_2)$,

$$\nabla\cdot\boldsymbol{E}_\circ = \nabla\cdot\mathcal{A}\boldsymbol{E}_\circ + \nabla\cdot\boldsymbol{D} = \nabla\cdot(\eta\boldsymbol{E}_\circ) + \nabla\cdot\boldsymbol{D}\,,$$

hence $\nabla\cdot(\varepsilon_r\boldsymbol{E}_\circ) = \nabla\cdot\boldsymbol{D} = 0$, and finally $\nabla\cdot\boldsymbol{E}_\circ = -\boldsymbol{\tau}\cdot\boldsymbol{E}_\circ \in L^2(\Omega^-)$. Let us check now that the couple $(\boldsymbol{E}_\circ, \gamma_n\boldsymbol{E}_\circ)$ satisfies the equations of $(\mathcal{E}_1)$. We have

$$\boldsymbol{E}_\circ = \nabla\mathcal{M}\boldsymbol{E}_\circ - \kappa^2\mathcal{N}\boldsymbol{E}_\circ + \boldsymbol{D}.$$

Since we know now that $\boldsymbol{E}_\circ \in H(\mathrm{div}, \Omega^-)$, we can use integration by parts and go back to $\nabla\mathcal{M}\boldsymbol{E}_\circ = \nabla\mathcal{Q}\boldsymbol{E}_\circ + \nabla\mathcal{L}(\gamma_n\boldsymbol{E}_\circ)$. Thus we have the first equation of $(\mathcal{E}_1)$,

$$\boldsymbol{E}_\circ = \nabla\mathcal{Q}\boldsymbol{E}_\circ + \nabla\mathcal{L}(\gamma_n\boldsymbol{E}_\circ) - \kappa^2\mathcal{N}\boldsymbol{E}_\circ + \boldsymbol{D},$$

and we obtain the second one by taking the normal trace on Γ. ■

Remark 9 We can use the same proof also in the case where $\boldsymbol{D} \in H(\mathrm{div}, \Omega^-)$ is arbitrary, not necessarily divergence free. We then have to modify the right hand side of $(\mathcal{E}_1)$ by replacing $\boldsymbol{D}$ with the function

$$\widetilde{\boldsymbol{D}} = \boldsymbol{D} - \nabla\mathcal{N}(\tfrac{1}{\varepsilon_r}\eta\nabla\cdot\boldsymbol{D}).$$

With this right hand side, $(\mathcal{E}_1)$ turns out to be equivalent to $(\mathcal{E}_2)$ (with right hand side $\boldsymbol{D}$). We then find $\nabla\cdot(\varepsilon_r\boldsymbol{E}) = \nabla\cdot\boldsymbol{D}$ in Ω^-. This relation is an immediate consequence of $(\mathcal{E}_2)$, but in order to deduce it from $(\mathcal{E}_1)$, we now have to see that the function $\widetilde{q} = \frac{1}{\varepsilon_r}(\nabla\cdot(\varepsilon_r\boldsymbol{E}) - \boldsymbol{D})$ satisfies the homogeneous scalar Lippmann-Schwinger equation (8).

Having shown that the problems $(\mathcal{P})$, $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ are all equivalent, we look now at the mapping properties of the integral operators. Their well-posedness will imply the one for the transmission problem, which is of course already well known [5]. A more important motivation for the analysis of the integral operators in $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ is the question of their suitability for numerical computations. The easier one is $(\mathcal{E}_1)$, because it involves only weakly singular integral operators whose mapping properties are well known:

Proposition 10 *Let the coefficient ε_r be in $C^1(\overline{\Omega^-})$ with $\varepsilon_r(x) \neq 0$ in $\overline{\Omega^-}$ and*

$$\varepsilon_r(x) \neq -1 \qquad \textit{on}\ \Gamma\,. \tag{10}$$

Then the matrix operator of the problem $(\mathcal{E}_1)$

$$\mathrm{A} = \begin{pmatrix} 1 - \nabla\mathcal{N}_{\boldsymbol{\tau}} + \kappa^2\mathcal{N}_\eta & -\nabla\mathcal{S}_\eta \\ \\ \kappa^2\gamma_n^-\mathcal{N}_\eta - \gamma_1^-\mathcal{N}_{\boldsymbol{\tau}} & 1 - \gamma_1^-\mathcal{S}_\eta \end{pmatrix}$$

from $L^2(\Omega^-)^3 \times H^{-\frac{1}{2}}(\Gamma)$ *to* $L^2(\Omega^-)^3 \times H^{-\frac{1}{2}}(\Gamma)$ *is Fredholm of index zero. If there is a point on* Γ *where* (10) *is not satisfied, then it is not Fredholm.*

Proof: The operators $\mathcal{N} : L^2(\Omega^-) \to H^2(\Omega^-)$ and $\mathcal{S} : H^{-\frac{1}{2}}(\Gamma) \to H^1(\Omega^-)$ are bounded. So $-\nabla\mathcal{N}_{\boldsymbol{\tau}} + \kappa^2\mathcal{N}_\eta$ is compact from $L^2(\Omega^-)^3$ to itself, $\kappa^2\gamma_n^-\mathcal{N}_\eta - \gamma_1^-\mathcal{N}_{\boldsymbol{\tau}}$ is compact from $L^2(\Omega^-)^3$ to $H^{-\frac{1}{2}}(\Gamma)$ and $\nabla\mathcal{S}_\eta$ is bounded from $H^{-\frac{1}{2}}(\Gamma)$ to $L^2(\Omega^-)^3$. For the operator $\gamma_1^-\mathcal{S}_\eta$ we use the jump relations and obtain for $x \in \Gamma$:

$$\gamma_1^-\mathcal{S}_\eta f(x) = \int_\Gamma \eta(y)\partial_{n_x}G_\kappa(x-y)f(y)\,ds(y) + \frac{1}{2}\eta(x)f(x).$$

Thus $(1 - \gamma_1^-\mathcal{S}_\eta)f = \frac{1}{2}(1+\varepsilon_r)f - \mathcal{T}(\eta f)$, where on our smooth boundary the operator $\mathcal{T}$ is bounded from $H^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$, so it is compact from $H^{-\frac{1}{2}}(\Gamma)$ to itself. With $\alpha = \frac{1}{2}(1+\varepsilon_r)$, the matrix A can therefore be written in the following form:

$$\mathrm{A} = \begin{pmatrix} 1 & \mathcal{B} \\ 0 & \alpha 1 \end{pmatrix} + \begin{pmatrix} \mathcal{K}_1 & 0 \\ \mathcal{K}_3 & \mathcal{K}_2 \end{pmatrix},$$

where $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ are compact operators and $\mathcal{B}$ is bounded. We see that if (10) is satisfied, then A is the sum of an invertible and a compact operator, hence Fredholm of index zero; and if $\alpha(x) = 0$ for some $x \in \Gamma$, then A is not Fredholm. ■

As a consequence of the equivalence theorems, Proposition 10 and the known uniqueness of the scattering problem, we obtain the following corollary:

Theorem 11 *Under the assumptions of problem* $(\mathcal{P})$*, the VIE* $(\mathcal{E}_2)$ *has a unique solution depending continuously on the data.*

More general questions of mapping properties of the strongly singular integral operator of the VIE $(\mathcal{E}_2)$ in L^2 or in $H(\mathrm{div})$, in particular its spectral theory, remain largely open. We have the following partial result:

Proposition 12 *Let* $\varepsilon_r \in C^1(\overline{\Omega^-})$ *and* $\eta = 1 - \varepsilon_r$.
(i) The operator

$$\mathcal{A} : \boldsymbol{E} \mapsto \nabla\mathcal{M}(\eta\boldsymbol{E}) - \kappa^2\mathcal{N}(\eta\boldsymbol{E})$$

is bounded from $L^2(\Omega^-)^3$ *to* $L^2(\Omega^-)^3$ *and from* $H(\mathrm{div},\Omega^-)$ *to* $H(\mathrm{div},\Omega^-)$.
(ii) If $\boldsymbol{E} \in L^2(\Omega^-)^3$ *is solution of*

$$(1 - \mathcal{A})\boldsymbol{E} = \boldsymbol{D}$$

with $\boldsymbol{D} \in H(\mathrm{div},\Omega^-)$*, then* $\boldsymbol{E} \in H(\mathrm{div},\Omega^-)$.
(iii) If $\varepsilon_r(x) \neq 0$ *in* $\overline{\Omega^-}$ *and* $\varepsilon_r(x) \neq -1$ *on* Γ*, then the nullspace of the operator* $1 - \mathcal{A}$ *in* $L^2(\Omega^-)^3$ *is finite dimensional, and the codimension of the closure in* $L^2(\Omega^-)^3$ *of the image of* $H(\mathrm{div},\Omega^-)$ *is finite.*

(iv) If $\varepsilon_r(x) \geq \varepsilon_1$ for all $x \in \Omega^-$, where ε_1 is a positive constant, then the operator $1-\mathcal{A}$ is a Fredholm operator of index zero in $L^2(\Omega^-)^3$, and it is strongly elliptic: There is a compact operator $\mathcal{K}_0$ and $c > 0$ such that for all $\boldsymbol{E} \in L^2(\Omega^-)^3$

$$\int_{\Omega^-} \overline{\boldsymbol{E}(x)} \cdot (1-\mathcal{A})\boldsymbol{E}(x)\, dx \geq c\, \|\boldsymbol{E}\|^2_{L^2(\Omega^-)} - \|\mathcal{K}_0 \boldsymbol{E}\|^2_{L^2(\Omega^-)} \,. \tag{11}$$

Proof: The assertions (i)–(iii) have been shown above. We only need to show the Gårding inequality (11). It is clear that up to a compact perturbation, the operator $\mathcal{A}$ coincides with $(1-\varepsilon_r)\mathcal{P}$, where the operator $\mathcal{P}$ is defined with the fundamental solution of the Laplace operator:

$$\mathcal{P}\boldsymbol{E}(x) = \nabla \int_{\Omega^-} \nabla_y G_0(x-y) \cdot \boldsymbol{E}(y)\, dy\,; \qquad G_0(x-y) = \frac{1}{4\pi|x-y|}\,.$$

The quadratic form $(\boldsymbol{E}, \mathcal{P}\boldsymbol{E}) = \displaystyle\int_{\Omega^-} \overline{\boldsymbol{E}(x)} \cdot \mathcal{P}\boldsymbol{E}(x)\, dx$ is the restriction to Ω^- of the corresponding quadratic form on $\mathbb{R}^3$. On $\mathbb{R}^3$, the operator $\mathcal{P}$ is a Fourier multiplier by the matrix function $\widehat{\mathcal{P}}(\boldsymbol{\xi}) = (\boldsymbol{\xi}\boldsymbol{\xi}^\top)/|\boldsymbol{\xi}|^2$. This is an orthogonal projector, and hence both $\mathcal{P}$ and $1-\mathcal{P}$ are positive semidefinite in $L^2(\Omega^-)^3$. It is also clear that the multiplication by a continuous function on $\overline{\Omega^-}$ commutes with $\mathcal{P}$ modulo compact operators on $L^2(\Omega^-)^3$. Define

$$\varepsilon_r^-(x) = \min\{1, \varepsilon_r(x)\}\,.$$

Then there holds up to a compact perturbation

$$\begin{aligned}
(\boldsymbol{E}, (1-\mathcal{A})\boldsymbol{E}) &\sim (\boldsymbol{E}, (1-(1-\varepsilon_r)\mathcal{P})\boldsymbol{E}) \\
&= (\boldsymbol{E}, \varepsilon_r^- \boldsymbol{E}) + (\boldsymbol{E}, (1-\varepsilon_r^-)(1-\mathcal{P})\boldsymbol{E}) + (\boldsymbol{E}, (\varepsilon_r - \varepsilon_r^-)\mathcal{P}\boldsymbol{E}) \\
&\sim (\boldsymbol{E}, \varepsilon_r^- \boldsymbol{E}) + (\boldsymbol{E}_1, (1-\mathcal{P})\boldsymbol{E}_1) + (\boldsymbol{E}_2, \mathcal{P}\boldsymbol{E}_2) \\
&\geq (\boldsymbol{E}, \varepsilon_r^- \boldsymbol{E})\,,
\end{aligned}$$

where $\boldsymbol{E}_1 = \sqrt{1-\varepsilon_r^-}\, \boldsymbol{E}$ and $\boldsymbol{E}_2 = \sqrt{\varepsilon_r - \varepsilon_r^-}\, \boldsymbol{E}$.
This shows (11) with $c = \min\{1, \varepsilon_1\}$. ■

5. Conclusion and perspectives

Under the realistic hypothesis of discontinuity of the electric permittivity across the boundary of a dielectric, we first derived two integral formulations: a volume integral equation and a coupled surface-volume system of integral equations. We also justified equivalence between the electromagnetic scattering problem and the two integral formulations. We established well-posedness for all the problems. The coupled surface-volume integral formulation was easy to analyze, because it involves only weakly singular integrals. The equivalence with the volume integral equation then gives results also for this strongly singular integral equation. Since the VIE is posed in L^2 and satisfies a Gårding inequality, it is suitable for numerical approximations using L^2-conforming finite elements, because any Galerkin method will lead to a stable discretization scheme.

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