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Developments in Boundary Element Methods for Time-Dependent Problems

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1 Introduction

Time dependence is an essential feature in many engineering applications that are modelled by partial differential equations and, eventually, by boundary integral equations. Boundary element methods (BEM) have been successfully applied to many such problems from fields like elastodynamics, fluid dynamics, or acoustics.

Such problems are frequently modelled by *hyperbolic* equations, and *elliptic* equations are obtained in the limit cases of stationary or time-harmonic problems or with the help of the Laplace transform. *Parabolic* problems can be obtained in their own right in problems of heat transfer or other diffusion problems or also as certain (large damping) limit cases: In electrodynamics, for instance, for small frequencies and large conductivities, the limiting parabolic equation is used to describe eddy current problems.

It has been known for a long time that parabolic and hyperbolic initial-boundary value problems allow a reduction to boundary integral equations in a way very similar to elliptic boundary value problems. But compared to the vast literature about the boundary integral equations for the elliptic case, covering both their theoretical properties and their numerical approximation, the mathematical treatment of the boundary integral equations for parabolic and hyperbolic problems is rather modest in volume. There have been, however, some significant developments in this area in recent years. In this article, we will try to give a survey over some of the main mathematical ideas involved. Given the fast growth of the field of BEM, both on the side of engineering applications and on the side of mathematical and numerical analysis, the list of references cannot, of course, be comprehensive.

2 Some highlights of the mathematical history of time-dependent BEM

In the wake of the big success of BEM for elliptic boundary value problems, there have been many experiences with time-dependent BEM from the engineering side at least since 1980. The books [2], [6] contain reports about such experiments for hyperbolic problems, and [6], [7], [23] for parabolic problems. We shall concentrate here on the description of *mathematical* ideas (while keeping in mind the needs of the applications).

In the field of boundary integral equations for *parabolic* problems, the classical subject is, just as for elliptic problems, the study of integral equations of the second kind. These are of Volterra type (and for more than one space dimension, of Fredholm-Volterra type), and their theory for the case of *smooth* boundaries is summarized in POGORZELSKI's works [26]. This theory works in L^p spaces and has as its basic result the fact that such a Fredholm-Volterra integral operator K has the typical Volterra property of being quasi-nilpotent, i. e., $\lambda + K$ is invertible for any $\lambda \neq 0$.

Recent generalizations of this classical theory have the common feature that they closely (and sometimes amazingly closely) parallel similar developments for the case of elliptic equations.

Thus PIRIOU's parabolic pseudodifferential operators [24, 25] treat parabolic equations as anisotropic elliptic equations. PIRIOU's calculus contains the complete theory (regularity and solvability in Sobolev spaces) for all the boundary integral operators arising in the treatment of the standard initial-boundary value problems for the heat equation, provided that the boundary is a closed smooth surface.

For closed non-smooth (Lipschitz) boundaries, the theory as well as a certain Nyström approximation method for the second-kind integral equations in L^p spaces was studied by DAHLBERG&VERCHOTA, BROWN, TORRES and others (see [1, 8, 13]). A collocation method for second-kind integral equations on certain non-smooth domains was studied in [12].

For elliptic problems, the *variational formulation* of boundary integral equations was the key point that allowed to import ideas from the field of finite element methods (FEM) into BEM. Thus the coercivity of the operator of the single layer potential, shown by NEDELEC&PLANCHARD [21] and HSIAO&WENDLAND [17] was essential in several respects: for the proof of convergence of *any* Galerkin approximation method (including recently studied methods based on modern ideas like multigrid, domain decomposition, h-p

or adaptive methods); for the possibility to allow open boundaries arising in crack, screen, or antenna problems; and also for the analysis of the coupling of boundary integral methods with other variational methods, e. g. variational inequalities or FEM–BEM coupling.

The coercivity of the single layer *heat* potential had been conjectured for some time, but when the correct result (see Theorem 5.2) was found in 1987 simultaneously by D. ARNOLD and the author, its essentially "elliptic" nature came as somewhat of a surprise. A proof of this result and its application to Galerkin methods on smooth surfaces was published by ARNOLD&NOON [3, 22]. Different methods of proof, generalizations of the result to other integral operators constituting the Calderón projector for the heat equation and to non-smooth surfaces as well as applications to various initial-boundary value problems and to the coupling with FEM were given by the author [11]. A generalization to another parabolic problem describing the Stokes flow, was given by HEBEKER&HSIAO [16].

A new and different development is LUBICH's "operational quadrature method" which is not based on the analogy with elliptic problems but rather takes the evolutional nature of the problem seriously. It allows to combine numerical methods for ordinary differential equations with standard BEM for the spatial (elliptic) part. LUBICH&SCHNEIDER [20] applied this technique to space-time integral equations for the heat equation, and LUBICH [19] showed how to apply it even to hyperbolic problems.

In the field of boundary integral equations for *hyperbolic problems*, apart from LUBICH's work just mentioned, variational methods are dominating, too. Here no (classical or recent) theory of second-kind integral equations seems to be available. BAMBERGER&HA DUONG [4] used a variational formulation of wave equation problems by J. L. Lions and local Fourier analysis to show coercivity of the single layer potential for the wave equation. Here the non-elliptic nature of the problem is felt in the loss of some regularity and also in the exponential growth of the stability constants with respect to time. This technique has been generalized to elastodynamics [5, 9] and to electrodynamics [27, 14, 28]. An independent development of variational methods for the space-time boundary integral equations in elastodynamics is described by KHUTORYANSKY [18, 29].

3 Notations

We will now study some of the above-mentioned ideas in closer detail. We consider only the simplest model problem of each type. Let $\Omega \subset \mathbb{R}^n$, $(n \geq 2)$,

be a domain with compact boundary Γ. The outer normal derivative is denoted by ∂_n. Let $T > 0$ be fixed. We denote by Q the space-time cylinder over Ω and Σ its lateral boundary:

$$Q = (0,T) \times \Omega \; ; \quad \Sigma = (0,T) \times \Gamma \; ; \quad \partial Q = (\{0\} \times \overline{\Omega}) \cup \Sigma \cup (\{T\} \times \overline{\Omega}) \, .$$

Elliptic problem (with frequency $\omega \in \mathbb{C}$):

$$\begin{array}{l} (\Delta + \omega^2) u = 0 \quad \text{in } \Omega \; ; \\ u = g \;\; \text{(Dirichlet)} \quad \text{or } \partial_n u = h \;\; \text{(Neumann)} \quad \text{on } \Gamma \; ; \\ \text{radiation condition at } \infty \, . \end{array} \tag{E}$$

Parabolic problem:

$$\begin{array}{l} (\partial_t - \Delta) u = 0 \quad \text{in } Q \; ; \\ u = g \;\; \text{(Dirichlet)} \quad \text{or } \partial_n u = h \;\; \text{(Neumann)} \quad \text{on } \Sigma \; ; \\ u = 0 \quad \text{for } t \leq 0 \, . \end{array} \tag{P}$$

Hyperbolic problem:

$$\begin{array}{l} (\partial_t^2 - \Delta) u = 0 \quad \text{in } Q \; ; \\ u = g \;\; \text{(Dirichlet)} \quad \text{or } \partial_n u = h \;\; \text{(Neumann)} \quad \text{on } \Sigma \; ; \\ u = 0 \quad \text{for } t \leq 0 \, . \end{array} \tag{H}$$

4 Representation formulas and integral operators

We derive boundary integral equations by a general method that is valid (under suitable hypotheses on the data, C^∞ will certainly suffice...) in the same way for all 3 types of problems. In fact, what counts for ($\mathscr{P}$) and ($\mathscr{H}$) is the fact that the boundary Σ is non-characteristic.

The first ingredient for a BEM is a fundamental solution G. In 3D we have, respectively:

$$G_\omega(x) = \frac{e^{i\omega|x|}}{4\pi|x|} \tag{$\mathscr{E}$}$$

$$G(t,x) = \begin{cases} (4\pi t)^{-3/2} e^{-\frac{|x|^2}{4t}} & (t \geq 0) \\ 0 & (t \leq 0) \end{cases} \tag{$\mathscr{P}$}$$

$$G(t,x) = \frac{1}{4\pi|x|}\,\delta(t-|x|)\,. \qquad (\mathscr{H})$$

From Green's formula, the following representation formulas follow for a solution u of the homogeneous partial differential equation and $x \notin \Gamma$; $[v]$ denotes the jump of v across Γ :

$$u(x) = \int_\Gamma \{\partial_{n(y)} G(x-y)[u(y)] - G(x-y)[\partial_n u(y)]\}\, d\sigma(y) \qquad (\mathscr{E})$$

$$u(t,x) = \int_0^t\!\!\int_\Gamma \{\partial_{n(y)} G(t-s,x-y)[u(s,y)] - G(t-s,x-y)[\partial_n u(y)]\}\, d\sigma(y) \quad (\mathscr{P})$$

$$\begin{aligned} u(t,x) &= \int_0^t\!\!\int_\Gamma \{\partial_{n(y)} G(t-s,x-y)[u(s,y)] - G(t-s,x-y)[\partial_n u(y)]\}\, d\sigma(y) \quad (\mathscr{H}) \\ &= \int_\Gamma \{\partial_{n(y)} \frac{1}{4\pi|x-y|}[u(t-|x-y|,y)] - \frac{\partial_{n(y)}|x-y|}{4\pi|x-y|}[\partial_t u(t-|x-y|,y)] \\ &\qquad - \frac{1}{4\pi|x-y|}[\partial_n u(t-|x-y|,y)]\, d\sigma(y)\,. \end{aligned}$$

Thus the representation in the parabolic case uses integration over the past portion of Σ, whereas in the hyperbolic case, only the intersection of the interior of the backward light cone with Σ is involved. In 3D, where Huyghens' principle is valid, the last formula shows that the integration can be restricted to Γ, giving a very simple representation by "retarded potentials".

We can write all 3 representation formulas in a unified way, thereby introducing the single layer potential $\mathscr{S}$ and the double layer potential $\mathscr{D}$:

$$u = \mathscr{D}([u]) - \mathscr{S}([\partial_n u])\,.$$

There hold the classical jump relations

$$\begin{array}{lcl@{\qquad}lcl} [\mathscr{D}v] &=& v \;; & [\partial_n \mathscr{D}v] &=& 0 \;; \\ [\mathscr{S}\varphi] &=& 0 \;; & [\partial_n \mathscr{S}\varphi] &=& -\varphi \;. \end{array}$$

It appears therefore natural to introduce the boundary operators from the one-sided traces on the exterior (Γ^+) and interior (Γ^-) of Γ:

$$\begin{array}{lcll} V &:=& \mathscr{S}\big|_\Gamma & \text{(single layer potential)} \\ K &:=& \frac{1}{2}(\mathscr{D}\big|_{\Gamma^+} + \mathscr{D}\big|_{\Gamma^-}) & \text{(double layer potential)} \\ K' &:=& \frac{1}{2}(\partial_n\mathscr{S}\big|_{\Gamma^+} + \partial_n\mathscr{S}\big|_{\Gamma^-}) & \text{(normal derivative of single layer potential)} \\ W &:=& -\partial_n\mathscr{D}\big|_\Gamma & \text{(normal derivative of double layer potential)} \end{array}$$

In the standard way, the jump relations together with these definitions lead to boundary integral equations for the Dirichlet and Neumann problems. Typically one has a choice of at least 4 equations for each problem: The first 2 equations come from taking the traces in the representation formula ("direct method"), the third one comes form a single layer representation

$$u = \mathscr{S}\psi \quad \text{with unknown } \psi$$

and the fourth one from a double layer representation

$$u = \mathscr{D}w \quad \text{with unknown } w \ :$$

For the exterior Dirichlet problem ($u\Big|_\Gamma = g$ given, $\partial_n u\Big|_\Gamma = \varphi$ unknown):

$$\begin{array}{lll} (D1) & V\varphi & = (-\frac{1}{2} + K)g \\ (D2) & (\frac{1}{2} + K')\varphi & = -Wg \\ (D3) & V\psi & = g \\ (D4) & (\frac{1}{2} + K)w & = g \end{array}$$

For the exterior Neumann problem ($u\Big|_\Gamma = g = v$ unknown, $\partial_n u\Big|_\Gamma = h$ given):

$$\begin{array}{lll} (N1) & (\frac{1}{2} - K)v & = -Vh \\ (N2) & Wv & = -(\frac{1}{2} + K')h \\ (N3) & (\frac{1}{2} - K')\psi & = -h \\ (N4) & Ww & = -h \end{array}$$

Remember that this formal derivation is rigorously valid for all 3 types of problems. One notes that second-kind and first-kind integral equations alternate nicely. For open surfaces, however, only the first-kind integral equations exist. The reason is that a boundary value problem on an open surface fixes not only a one-sided trace but also the jump of the solution; and therefore the representation formula coincides with a single layer representation for the Dirichlet problem and with a double layer potential representation for the Neumann problem.

For reasons mentioned above, we will not treat the second-kind boundary integral equations in detail here. Suffice it to say that the key observation in the parabolic case is the fact that for smooth Γ, the operator norm in $L^p(\Sigma)$ of the weakly singular operator K tends to 0 as $T \to 0$. This implies that $\frac{1}{2} \pm K$ and $\frac{1}{2} \pm K'$ are isomorphisms in L^p (and also in C^m), first for small T and then by iteration for all T. If Γ has corners, this argument breaks down, and quite different methods, including variational arguments, have to be used.

5 First-kind integral operators: Green's formula

In the following, we restrict the presentation to the single layer potential operator V. We emphasize, however, that a completely analogous theory is available for the hypersingular operator W.

The variational methods for the first-kind integral operators are based on the first Green formula which gives, together with the jump relations, a formula valid again for all 3 types of equations: If φ and ψ are given on Γ or Σ, satisfy a finite number of conditions guaranteeing the convergence of the integrals on the right hand side of (5.1) and

$$u = \mathscr{S}\varphi, \qquad v = \mathscr{S}\psi,$$

then

$$\int_\Gamma \varphi V\psi \, d\sigma = \int_{\mathbb{R}^n\setminus\Gamma} \{\nabla u \cdot \nabla v + u\Delta v\} \, dx . \tag{5.1}$$

5.1 $(\mathscr{E})$

For the elliptic case, we obtain ($<\cdot,\cdot>_\Gamma$ denotes L^2 duality on Γ);

$$<\varphi, V\varphi>_\Gamma = \int_{\mathbb{R}^n\setminus\Gamma} (|\nabla u|^2 - \omega^2|u|^2) \, dx .$$

This gives the following theorem (see [10]) that serves as a model for the other two types:

Theorem 5.1 *Let Γ be a bounded Lipschitz surface, open or closed. $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ denote the usual Sobolev spaces, and $\tilde{H}^{-1/2}(\Gamma)$ for an open surface is the dual of $H^{1/2}(\Gamma)$. Then*
(i) For $\omega = 0$, $n \geq 3$: $V : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an isomorphism, and there is an $\alpha > 0$ such that

$$<\varphi, V\varphi>_\Gamma \geq \alpha \|\varphi\|^2_{\tilde{H}^{-1/2}(\Gamma)} .$$

(ii) For any ω and n, there is an $\alpha > 0$ and a compact quadratic form k on $\tilde{H}^{-1/2}(\Gamma)$ such that

$$\mathrm{Re}\, <\varphi, V\varphi>_\Gamma \geq \alpha \|\varphi\|^2_{\tilde{H}^{-1/2}(\Gamma)} - k(\varphi) .$$

(iii) If ω is not an interior or exterior eigenfrequency, then V is an isomorphism, and every Galerkin method in $\tilde{H}^{-1/2}(\Gamma)$ for the equation $V\psi = g$ converges.

5.2 $(\mathscr{P})$

For the parabolic case, integration over t gives

$$\begin{aligned} <\varphi, V\varphi>_\Sigma &= \int_0^T \int_{\mathbb{R}^n \setminus \Gamma} \{|\nabla_x u(t,x)|^2 + \partial_t u \overline{u}\}\, dx\, dt \\ &= \iint |\nabla_x u(t,x)|^2\, dx\, dt + \frac{1}{2}\int_{\mathbb{R}^n} |u(T,x)|^2\, dx\,. \end{aligned}$$

The positivity of the quadratic form associated to the operator V is evident. What is less evident is the nature of the energy norm for V, however. It turns out [3, 11] that one has to consider anisotropic Sobolev spaces of the following form

$$\tilde{H}_0^{r,s}(\Sigma) = L^2(0,T;\, \tilde{H}^r(\Gamma)) \cap H_0^s(0,T;\, L^2(\Gamma))\,.$$

The index 0 indicates that zero initial conditions at $t = 0$ are incorporated. The optional $\tilde{}$ means zero boundary values on the boundary of the (open) manifold Γ. One has the following theorem which is actually simpler than its elliptic counterpart.

Theorem 5.2 *Let Γ be a bounded Lipschitz surface, open or closed, $n \geq 2$. (i) $V : \tilde{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H_0^{r,s}(\Sigma)$ is an isomorphism, and there is an $\alpha > 0$ such that*

$$<\varphi, V\varphi>_\Sigma \geq \alpha \|\varphi\|^2_{-\frac{1}{2},-\frac{1}{4}}\,.$$

(ii) Every Galerkin method in $\tilde{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ for the equation $V\psi = g$ converges. The Galerkin matrices have positive definite symmetric part. Typical error estimates are of the form

$$\|\varphi - \varphi_{h,k}\|_{-\frac{1}{2},-\frac{1}{4}} \leq C\,(h^{r+\frac{1}{2}} + k^{(r+\frac{1}{2})/2})\|\varphi\|_{r,\frac{r}{2}}\,,$$

if $\varphi_{h,k}$ is the Galerkin solution in a tensor product space of splines of mesh-size k in time and finite elements of mesh-size h in space.

5.3 $(\mathscr{H})$

For the wave equation, choosing $\varphi = \overline{\psi}$ in the Green formula (5.1) does not give a positive definite expression. Instead, one can choose $\varphi = \overline{\partial_t \psi}$. This corresponds to the usual procedure in the weak formulation of the wave equation,

and it gives

$$\begin{aligned} <\partial_t\varphi, V\varphi>_\Sigma &= \int_0^T \int_{\mathbb{R}^n\setminus\Gamma} \{\partial_t \nabla_x \overline{u} \cdot \nabla_x u + \overline{u}\partial_t^2 u\}\, dx\, dt \\ &= \frac{1}{2}\int_{\mathbb{R}^n\setminus\Gamma} \{|\nabla_x u(T,x)|^2 + |\partial_t u(T,x)|^2\}\, dx\,. \end{aligned}$$

Once again, as in the elliptic case, this shows the close relation of the operator V with the total energy of the system. In order to obtain a norm ($H^1(Q)$) on the right hand side, one can integrate a second time over t. But in any case, here the bilinear form $<\partial_t\varphi, V\varphi>_\Sigma$ will not be bounded in the same norm where its real part is positive. So there will be a loss of regularity, and any error estimate has to use two different norms. No "natural" energy space for the operator V presents itself.

6 First-kind integral operators: Fourier analysis

A closer view of what is going on can be obtained using space-time Fourier transformation. For this, one has to assume that Γ is flat, i. e. a subset of $\mathbb{R}^{n-1}$. Then all the operators are convolutions and as such are represented by multiplication operators in Fourier space. If Γ is not flat but smooth, then the results for the flat case describe the principal part of the operators. To construct a complete analysis, one has to consider lower order terms coming from coordinate transformations and localizations. Whereas this is a well-known technique in the elliptic and parabolic cases, namely part of the calculus of pseudodifferential operators, it has so far prevented the construction of a completely satisfactory theory for the hyperbolic case.

We denote the dual variables to (t,x) by (ω,ξ), and x' and ξ' are the variables related to $\Gamma \subset \mathbb{R}^{n-1}$. It is then easily seen that the form of the single layer potential is

$$\widehat{V\psi}(\xi') = \frac{1}{2}(|\xi'|^2 - \omega^2)^{-\frac{1}{2}}\hat{\psi}(\xi') \qquad (\mathscr{E})$$

$$\widehat{V\psi}(\omega,\xi') = \frac{1}{2}(|\xi'|^2 + i\omega)^{-\frac{1}{2}}\hat{\psi}(\omega,\xi') \qquad (\mathscr{P})$$

$$\widehat{V\psi}(\omega,\xi') = \frac{1}{2}(|\xi'|^2 - \omega^2)^{-\frac{1}{2}}\hat{\psi}(\omega,\xi') \qquad (\mathscr{H})$$

Note that $(\mathscr{E})$ and $(\mathscr{H})$ differ only in the role of ω: For $(\mathscr{E})$ it is a fixed parameter, for $(\mathscr{H})$ it is one of the variables, and this is crucial in the application of Parseval's formula for $<\varphi, V\varphi>$.

6.1 $(\mathscr{E})$

For the elliptic case, the preceding formula implies Theorem 5.1: If $\omega = 0$, then the function $\frac{1}{2}|\xi'|^{-1}$ is positive and for large $|\xi'|$ equivalent to $(1+|\xi'|^2)^{-1/2}$, the Fourier weight defining the Sobolev space $H^{-1/2}(\Gamma)$. If $\omega \neq 0$, then the principal part (as $|\xi'| \to \infty$) is still $\frac{1}{2}|\xi'|^{-1}$, so only a compact perturbation is added. There is an additional observation by HA DUONG [15]: If ω is real, then $\frac{1}{2}(|\xi'|^2 - \omega^2)^{-\frac{1}{2}}$ is either positive or imaginary, so its real part is positive except on the bounded set $|\xi'| \leq |\omega|$. This implies

Proposition 6.1 *Let $\omega^2 > 0$, Γ flat,* $\operatorname{supp}\varphi$ *compact. Then there is an $\alpha(\omega) > 0$ such that*

$$\operatorname{Re} <\varphi, V\varphi>_\Gamma \geq \alpha(\omega)\, \|\varphi\|^2_{\tilde{H}^{-1/2}}\,.$$

The work of transforming this estimate into error estimates for the BEM in the hyperbolic case still has to be done.

6.2 $(\mathscr{P})$

For the parabolic case, the symbol of the single layer potential,

$$\sigma_V(\omega, \xi') = \frac{1}{2}(|\xi'|^2 + i\omega)^{-\frac{1}{2}}$$

has again positive real part. In addition, it is sectorial:

$$|\arg \sigma_V(\omega, \xi')| \leq \frac{\pi}{4}\,.$$

This has the consequence that its real part and absolute value are equivalent (an "elliptic" situation):

$$C_1 \left||\xi'|^2 + i\omega\right|^{-\frac{1}{2}} \leq \operatorname{Re}\sigma_V(\omega, \xi') \leq C_2 \left||\xi'|^2 + i\omega\right|^{-\frac{1}{2}}\,.$$

In addition, for large $|\xi'|^2 + |\omega|$, this is equivalent to $((1+|\xi'|^2) + |\omega|)^{-1/2}$, the Fourier weight defining the space $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. This explains Theorem 5.2. It also shows clearly the difference to the heat operator $\partial_t - \Delta$ itself: The symbol of the latter is $|\xi|^2 + i\omega$, and the real part and absolute value of this function are not equivalent.

6.3 $(\mathscr{H})$

In the hyperbolic case, the symbol σ_V does not have positive real part. Instead, one has to multiply it by $i\overline{\omega}$ and to use a complex frequency $\omega = \omega_R + i\omega_I$ with ω_I fixed. Then one gets

$$\mathrm{Re}\left(i\overline{\omega}(|\xi'|^2 - \omega^2)^{\frac{1}{2}}\right) \geq \frac{\omega_I}{2}(|\xi'|^2 + |\omega|^2)^{\frac{1}{2}}$$

and similar estimates given by BAMBERGER&HA DUONG [4]. One introduces another class of anisotropic Sobolev spaces of the form

$$H^{s,r}(\mathbb{R}\times\Gamma) = \{u \mid u, \partial_t^r u \in H^s(\mathbb{R}\times\Gamma)\}$$

with the norm

$$\|u\|_{r,s,\omega_I} = \int\limits_{\mathrm{Im}\,\omega=\omega_I} \int_{\mathbb{R}^{n-1}} |\omega|^{2r}(|\xi'|^2 + |\omega|^2)^s |\hat{u}(\omega,\xi')|^2 \, d\xi' \, d\omega \,.$$

We give one example of a theorem obtained in this way.

Theorem 6.2 *Let Γ be bounded and smooth, $r, s \in \mathbb{R}$. Then*
(i) $V : \tilde{H}_0^{s,r+\frac{1}{2}}(\Sigma) \to H_0^{s+1,r}(\Sigma)$ and $V^{-1} : H^{s+1,r+1}(\Sigma) \to \tilde{H}_0^{s,r}(\Sigma)$ are continuous.
(ii) Let $\omega_I > 0$ and the bilinear form $a(\varphi,\psi)$ be defined by

$$a(\varphi,\psi) = \int_0^\infty e^{-2\omega_I t} \int_\Gamma (V\varphi)(t,x)\, \overline{\partial_t \psi}(t,x)\, d\sigma(x)\, dt \,.$$

Then there is an $\alpha > 0$ such that

$$\mathrm{Re}\, a(\varphi,\varphi) \geq \alpha\, \omega_I \, \|\varphi\|^2_{-\frac{1}{2},0,\omega_I} \,.$$

(iii) The Galerkin matrices for the scheme: Find $\varphi_N \in X_N$ such that

$$a(\varphi_N,\psi) = <g, \partial_t\psi>_\Sigma \quad \forall \psi \in X_N$$

have positive definite hermitian part, and there is an error estimate

$$\|\varphi - \varphi_N\|_{-\frac{1}{2},0,\omega_I} \leq C\, \omega_I^{-\frac{1}{2}} \inf_{\psi\in X_N} \|\varphi - \psi\|_{-\frac{1}{2},1,\omega_I} \,.$$

7 A time-stepping method

The Galerkin methods for parabolic and hyperbolic problems studied in the previous sections are *global in time*: They use the boundary data for $0 \leq t \leq T$ to compute the unknowns on the boundary and hence the solution u also for $t \in [0,T]$ in one step. Thus, in general, the Volterra convolution structure (causality) of the differential and integral equations will be lost by discretization. Only in special cases, this structure is conserved on the discrete level. If, for example, for the single layer heat potential, the trial functions are piecewise constant in time, then the Galerkin matrix will have the form

$$\begin{pmatrix} B_0 & & & 0 \\ B_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ B_{n_t} & \cdots & B_1 & B_0 \end{pmatrix} \tag{7.1}$$

Here the blocks B_j are $n_x \times n_x$ matrices ($n_x =$ number of degrees of freedom of the space discretization; $n_t =$ number of time steps). Thus only B_0 has to be inverted, and in order to increase n_t by one, only one new matrix B_{n_t+1} has to be computed. If, however, the order of approximation in time is increased, then the matrix will have more and more blocks above the diagonal and therefore lose the discrete causal structure. For piecewise linear approximation in time with the usual hat function basis, for example, the matrix will have the form

$$\begin{pmatrix} B_0 & B_{-1} & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & B_{-1} \\ B_{n_t} & \cdots & \cdots & B_0 \end{pmatrix}$$

This problem vanishes if one replaces the space-time Galerkin boundary element approximation by a different method, the recently developped "operational quadrature" method of LUBICH [19, 20]. We will not spoil the reader's pleasure of reading these two papers by describing their contents in detail; we shall rather give an indication of the basic idea.

In this method, one discretizes not the space-time operator with its kernel as given above in section 4, but rather its Laplace transform with respect to time. This kernel corresponds to an elliptic problem, and for the space discretization one can choose any suitable BEM for elliptic problems, for instance collocation instead of Galerkin methods. For the time discretization, one chooses a

discretization scheme for ordinary differential equations. This scheme has to satisfy certain stability conditions. If it is an explicit linear multistep method, then the resulting matrix will be of the block triangular form 7.1, although the method can be of higher order in time.

More precisely, the operational quadrature method considers an operator-valued convolution operator

$$g \mapsto \int_0^t k(t-\tau)\, g(\tau)\, d\tau =: K(\partial_t)\, g\,.$$

The basic object is the Laplace-transformed kernel $K(s) = \int_0^\infty e^{-st} k(t)\, dt$. With a linear multistep method for $y' = f(t,y)$,

$$a_0 y_n + a_1 y_{n-1} + \cdots + a_k y_{n-k} = h\,(b_0 f_n + \cdots + b_k f_{n-k})$$

and its characteristic function,

$$\delta(\zeta) = (a_0 + a_1\zeta + \cdots + a_k\zeta^k)/(b_0 + b_1\zeta + \cdots + b_k\zeta^k)\,,$$

one constructs the following approximation of $K(\partial_t)g$:

$$K(\partial_t^h)g(t) = \sum_{j\geq 0} \omega_j g(t - jh)\,,$$

where the ω_j are the Taylor coefficients of $K(\delta(\zeta)/h)$ at $\zeta = 0$:

$$K\Big(\frac{\delta(\zeta)}{h}\Big) = \sum_{j\geq 0} \omega_j \zeta^j\,.$$

This discretization has the following decisive properties:

$$K_1(\partial_t^h) \cdot K_2(\partial_t^h) = (K_1 \cdot K_2)(\partial_t^h)\,;$$

if $y_j = (K(\partial_t^h)g)(jh)$ and $Y(\zeta) = \sum y_j\zeta^j$, $G(\zeta) = \sum g_j\zeta^j$ then

$$Y(\zeta) = K\Big(\frac{\delta(\zeta)}{h}\Big)\, G(\zeta)\,.$$

In the application to time-dependent BEM, one thinks of K as the inverse of the space discretization. $K(\partial_t^h)$ is then an approximation of the inverse of the space-time integral operator in question. The coefficients are therefore the result of an elliptic BEM performed for a certain number of complex frequencies $\frac{\delta(\zeta)}{h}$, where ζ runs through the nodes of some quadrature rule on a

small circle in the complex plane. This method promises to be very efficient, in particular since it allows to combine some of the recently developped sophisticated BEM for elliptic problems with well-known high order methods for ordinary differential equations and fast methods for the computation of Fourier coefficients. How it competes, for example, with very simple direct space-time methods using retarded potentials for the 3D wave equation, remains to be seen, however.

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