Crack singularities for general elliptic systems

MARTIN COSTABEL and MONIQUE DAUGE

Abstract. *We consider general homogeneous Agmon-Douglis-Nirenberg elliptic systems with constant coefficients complemented by the same set of boundary conditions on both sides of a crack in a two-dimensional domain. We prove that the singular functions expressed in polar coordinates* (r,θ) *near the crack tip all have the form* $r^{\frac{1}{2}+k}\varphi(\theta)$ *with* $k \geq 0$ *integer, with the possible exception of a finite number of singularities of the form* $r^k \log r\,\varphi(\theta)$. *We also prove results about singularities in the case when the boundary conditions on the two sides of the crack are not the same, and in particular in mixed Dirichlet-Neumann boundary value problems for strongly coercive systems: in the latter case, we prove that the exponents of singularity have the form* $\frac{1}{4} + i\eta + \frac{k}{2}$ *with real* η *and integer* k. *This is valid for general anisotropic elasticity too.*

Introduction

It is well known that the solutions of the Neumann problem for the Laplace operator or the Lamé system have strong singularities near a crack tip. In general they do not belong to $H^{\frac{3}{2}}$ due to the presence of a singular function of the form $r^{\frac{1}{2}}\varphi(\theta)$, with (r,θ) the polar coordinates centered at the crack tip.

We prove in this paper that such a situation is still valid under very general hypotheses. Let us recall that the general theory of elliptic boundary value problems near conical singularities, *cf* KONDRAT'EV [10] and GRISVARD [8], yields that singular parts in the solutions are present at the crack tip and always have the form

$$r^{\lambda}\sum_q \log^q r\varphi_q(\theta),$$

where the λ are complex numbers, called the *singularity exponents*. We prove that for general elliptic systems in the sense of AGMON, DOUGLIS & NIRENBERG [1] with the *same* boundary conditions on both sides of the crack, the singularity exponents all have the form $\frac{1}{2}+k$ with integer k, with the possible exception of a finite number of integer numbers. In particular for systems of elasticity with general material law, we find again the result proved by KOZLOV & MAZ'YA [12] and by DUDUCHAVA & WENDLAND [7].

Extending our investigation to the more general situation where we may have two different sets of boundary conditions on the two sides on the crack, we prove that in a fairly general situation the exponents are distributed with a period equal to $\frac{1}{2}$: this fact is not in contradiction with the situation when the boundary conditions are the same on both sides of the crack: in the latter situation the integers also appear as primary exponents of singularities, but at the end these integers do not play any role because the associated "singular functions" are nothing but polynomials.

When the operator is a self-adjoint strongly coercive system and the boundary conditions are mixed Dirichlet – Neumann, coupling our analysis with an argument inspired by the paper [11] by KOZLOV & MAZ'YA, we obtain that the singularity exponents have the form $\frac{1}{4}+i\eta+k$ with real η and integer k. This is also valid for systems of elasticity with general material law, where our result gives back the result of DUDUCHAVA & NATROSHVILI [6].

While the method of [7] and [6] is based on a pseudo-differential reduction to the boundary and a partial Wiener-Hopf factorization, our method of investigation is rather simple and explicit, based on a way of computing *characteristic matrices*, i.e. finite dimensional matrices $\mathscr{N}(\lambda)$ which are singular for the λ which are the singularity exponents. This construction of characteristic matrices was introduced in our work [3].

Here we concentrate on the *main singularities*, i.e. those coming directly from the poles of the inverse of the Mellin symbol, see (1.5), associated with the principal part of the boundary value problem with coefficients frozen at the crack tip. These main singularities generate directly the solution asymptotics near the crack tip when the boundary value problem is homogeneous with constant coefficients. When the boundary value problem is inhomogeneous, the solution asymptotics is a linear combination of these main singularities and of "shadow" singularities whose exponents are simply the main exponents shifted by positive integers.

Moreover, when the domain is three-dimensional and is exterior to a screen region, the singularities are still generated by the main singularities at a crack tip for associated two-dimensional problems and our present study allows a description of the three-dimension edge singularities, *cf* a forthcoming work.

Our paper is organized as follows: in §1 we set the boundary value problems and define the singularity spaces and singularity exponents; in §2 we give Cauchy residue formulas for these singularity spaces which allow in §3 the introduction and the study of the characteristic matrix $\mathscr{N}(\lambda)$. In §4 and 5 we deduce from the previous formulas the above mentioned results in the general case and in the case when the boundary conditions are the same on each side of the crack. In §6 and 7, we investigate mixed Dirichlet-Neumann boundary conditions for general strongly coercive self-adjoint systems, and for elasticity operators. We conclude our paper in §8 by regularity results for solutions of the above mentioned problems.

1 Elliptic systems in a domain with crack

1.a A multi-degree elliptic boundary value problem

Let $\mathscr{C} := \mathbb{R}^2 \setminus \Gamma$ be the infinite bi-dimensional model region with crack. Here Γ is a half line. We can assume that in a suitable system of coordinates (x_1, x_2)

$$\Gamma = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid \ x_1 > 0, \quad x_2 = 0\big\}.$$

Let (r, θ) be the standard polar coordinates such that $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$.

Let $L(\partial_1, \partial_2)$ be a $N\times N$ *properly elliptic system* in the sense of AGMON-DOUGLIS-NIRENBERG [1], homogeneous with constant coefficients. This means that there exist two sequences of positive integers $(\sigma_1, \dots, \sigma_N)$ and $(\tau_1, \dots, \tau_N)$ such that the order of the operator coefficient L_{jk} of L is $\sigma_j - \tau_k$. The assumption that L is homogeneous means that L_{jk} has no lower order term. If $\sigma_j - \tau_k$ is < 0, it is understood that $L_{jk} = 0$.

Let m be the half-sum of degrees $(\sigma_1 - \tau_1) + \dots + (\sigma_N - \tau_N)$. The proper ellipticity means that m is integer and that for every pair of independent real vectors $\Xi = (\xi_1, \xi_2)$ and $\Xi' = (\xi_1', \xi_2')$ the polynomial in t, $\det L(\Xi + t\Xi')$ has m roots with positive imaginary part and m roots with negative imaginary part.

Let $B = (B_1, \dots, B_m)$ and $C = (C_1, \dots, C_m)$ two systems of *complementing boundary conditions* for L on Γ in the sense of [1]. Every operator B_h has the operator coefficients B_{hk}, $k = 1, \dots, N$ and there exists a sequence of positive integers $(\rho_1^B, \dots, \rho_m^B)$ such that the degree of B_{hk} is $\rho_h - \tau_k$. Similarly the coefficients C_{hk} of the operator C_h has degree $\rho_h^C - \tau_k$, with positive integers $(\rho_1^C, \dots, \rho_m^C)$. Like the operators L_{jk}, all these boundary operators are supposed to have no lower order term.

Let us denote by γ_+ and γ_- the trace operators on Γ from the upper and lower half plane respectively. In the present paper we are interested in the properties near the crack tip 0 of any vector function $u = (u_1, \dots, u_N)$ solution of the following boundary value problem

$$\left\{ \begin{array}{rcll} Lu & = & f & \text{in } \mathscr{C}, \\ \gamma_+(Bu) & = & g_+ & \text{on } \Gamma, \\ \gamma_-(Cu) & = & g_- & \text{on } \Gamma, \end{array} \right. \tag{1.1}$$

with the data $f = (f_1, \dots, f_N)$, $g_+ = (g_{+,1}, \dots, g_{+,m})$ and $g_- = (g_{-,1}, \dots, g_{-,m})$ smooth enough.

This problem appears as a particular case in the general theory of elliptic boundary value problems in domains with conical singularities as investigated by KONDRAT'EV in [10]. The main result coming from the general theory is the splitting of any solution

u into a regular part $u_{\rm reg}$ and a linear combination of a finite number of *singular functions* $u_{\rm sing}^{[j]}$ which only depend on the geometry of the domain and the coefficients of the operators.

From explicit formulas relating to classical operators such as the Laplacian and the Lamé system, *cf* GRISVARD [8, 9], it is well known that in the presence of a crack (a plane angle with opening 2π) the Dirichlet and Neumann problems for the above mentioned operators have all their singular functions of the form $r^{\frac{1}{2}+k}\varphi(\theta)$ with (r,θ) the polar coordinates such that $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$.

Several works are devoted to different boundary value problems for the elasticity system with general material law: let us quote KOZLOV & MAZ'YA [12] and DUDUCHAVA & WENDLAND [7] who prove that the Dirichlet or Neumann boundary conditions on *both sides* of the crack still yield singularity exponents equal to $\frac{1}{2}+k$ and DUDUCHAVA & NATROSHVILI [6] who prove that the mixed Dirichlet – Neumann problem has $\frac{1}{4}+i\eta+\frac{k}{2}$ as singularity exponents (with $\eta\in\mathbb{R}$ and $k\in\mathbb{N}$).

Our aim is the investigation of these singular functions $u_{\rm sing}^{[j]}$ at the crack tip for the most general elliptic system in the sense of [1]. Our method is also a simpler alternative to the Wiener-Hopf method used in [7, 6].

1.b Singularity spaces

Let us now recall the description of singularities introduced in [5, 3]. Singularities are in close relation with *pseudo-homogeneous solutions* of the above problem (1.1) with zero data $f=0$, $g_+=g_-=0$. Let us introduce the relevant spaces of pseudo-homogeneous vector functions in relation with the multi-degree of problem (1.1). Let λ be a complex number. The space $S^\lambda(\mathscr{C})$ associated with λ is defined as follows in polar coordinates (r,θ)

$$S^\lambda(\mathscr{C}) = \big\{ u=(u_1,\ldots,u_N)\in\mathscr{D}'(\mathscr{C})^N \;\mid\; u_k = r^{\lambda-\tau_k}\sum_{q=0}^{Q}\log^q r\,\varphi_k^q(\theta)\big\}. \tag{1.2}$$

In the above definition Q is an arbitrary integer and the angular functions φ_k^q belong to $\mathscr{C}^\infty([0,2\pi])$. Let $S_0^\lambda(\mathscr{C})$ be the subspace of homogeneous functions in $S^\lambda(\mathscr{C})$:

$$S_0^\lambda(\mathscr{C}) = \big\{ u=(u_1,\ldots,u_N)\in\mathscr{D}'(\mathscr{C})^N \;\mid\; u_k = r^{\lambda-\tau_k}\,\varphi_k(\theta)\big\}. \tag{1.3}$$

The first singularity space associated with problem (1.1) is

$$\mathscr{X}^\lambda(\mathscr{C}) = \big\{ u\in S^\lambda(\mathscr{C}) \;\mid\; u \text{ solution of (1.1) with } f=0,\ g_+=g_-=0\,\big\}. \tag{1.4}$$

Of course, only the cases when this space is not reduced to $\{0\}$ correspond to the presence of singularities. The corresponding λ are those for which the *Mellin symbol* $M(\lambda)$ is not invertible. They are called *"singular exponents"*.

We recall that the Mellin symbol M of problem (1.1) is determined by the effect of the operator of problem (1.1) on elements of $S_0^\lambda(\mathscr{C})$ and is defined for any complex number λ as

$$M(\lambda) : \mathscr{C}^\infty([0,2\pi])^N \ni \varphi \longmapsto (\psi, \chi_+, \chi_-) \in \mathscr{C}^\infty([0,2\pi])^N \times \mathbb{C}^m \times \mathbb{C}^m$$

with

$$\begin{cases} \sum_k L_{jk}(r^{\lambda-\tau_k}\varphi_k) &= r^{\lambda-\sigma_j}\psi_j \quad \text{in } \mathscr{C}, \quad j = 1, \dots, N, \\ \gamma_+\big(\sum_k B_{h,k}(r^{\lambda-\tau_k}\varphi_k)\big) &= r^{\lambda-\rho_h^B}\chi_{+,h} \quad \text{on } \Gamma, \quad h = 1, \dots, m, \\ \gamma_-\big(\sum_k C_{h,k}(r^{\lambda-\tau_k}\varphi_k)\big) &= r^{\lambda-\rho_h^C}\chi_{-,h} \quad \text{on } \Gamma, \quad h = 1, \dots, m. \end{cases} \tag{1.5}$$

According to the general theory [2, 10], the inverse $M(\lambda)^{-1}$ of the symbol M is a meromorphic operator valued function.

Then the space $\mathscr{X}^\lambda(\mathscr{C})$ in (1.4) is not reduced to $\{0\}$ if and only if λ is a pole of the inverse of the Mellin symbol $M(\lambda)$: for any pole λ_0 of M^{-1} there holds

$$\mathscr{X}^{\lambda_0}(\mathscr{C}) = \Big\{ \frac{1}{2i\pi}\operatorname{diag}\big(r^{-\tau_k}\big) \int_{\gamma(\lambda_0)} r^\lambda M(\lambda)^{-1}\Psi(\lambda)\, d\lambda \mid \\ \Psi \in \mathfrak{A}[\lambda] \otimes \mathscr{C}^\infty([0,2\pi])^N \times \mathbb{C}^m \times \mathbb{C}^m \Big\}, \tag{1.6}$$

with a closed contour $\gamma(\lambda_0)$ surrounding λ_0 (and no other pole of M^{-1}) and the space of analytic functions $\mathfrak{A}[\lambda]$.

Moreover the non-zero elements of the spaces $\mathscr{X}^\lambda(\mathscr{C})$ are the singular functions in the basic weighted Sobolev spaces of KONDRAT'EV's theory [10].

If ordinary Sobolev spaces are used to describe the regularity of data and solutions, we have to take polynomials into account and define other singularity spaces for any integer $\lambda \in \mathbb{N}$, see [5]. We will come back to this in section 5 devoted to the situation where the boundary conditions imposed on both sides of the crack are the same ($B = C$).

2 Residue formulas for singularities

We are going to study the spaces $\mathscr{X}^\lambda(\mathscr{C})$. For this we first recall a few notations and results of [3].

2.a Residue representation for the singularity spaces

The first result is that there holds a residue representation of $\mathscr{X}(\lambda_0)$ like (1.6) involving only finite dimensional objects. Let us introduce first

$$\mathfrak{W}(\lambda) = \big\{ u \in S_0^\lambda(\mathscr{C}) \mid \; Lu = 0 \big\}. \tag{2.1}$$

There holds [3, Th.2.1]

Theorem 2.1 *For all complex number λ, the dimension of the solution space $\mathfrak{W}(\lambda)$ is equal to $2m$ and there exist $2m$ analytic (with respect to λ) vector functions $w_\ell^\pm(\lambda)$, $\ell = 1,\ldots,m$, which form a basis of $\mathfrak{W}(\lambda)$ for any non-integer λ. The two subspaces $\mathfrak{W}^+(\lambda)$ and $\mathfrak{W}^-(\lambda)$ of dimension m generated by $w_\ell^\pm(\lambda)$ have a special structure* cf Lemma 2.4.

We introduce the following two matrices

Definition 2.2 *a.* $W(\lambda)$ is the $N\times 2m$ matrix formed with the two $N\times m$ blocks $W^\pm(\lambda)$ with coefficients $W^\pm_{k\ell}(\lambda) = w^\pm_{\ell k}(\lambda)$, where $w^\pm_{\ell 1}(\lambda)$, ..., $w^\pm_{\ell N}(\lambda)$ are the N components of the basis vector functions in Theorem 2.1:

$$W(\lambda) = \begin{pmatrix} W^+(\lambda) & W^-(\lambda)\end{pmatrix}. \tag{2.2}$$

b. $\mathscr{N}(\lambda)$ is called the *characteristic matrix* and is defined as the 2×2 block matrix with the $m\times m$ blocks

$$\mathscr{N}(\lambda) = \begin{pmatrix} \mathscr{B}^+(\lambda) & \mathscr{B}^-(\lambda) \\ \mathscr{C}^+(\lambda) & \mathscr{C}^-(\lambda)\end{pmatrix}, \tag{2.3}$$

where $\mathscr{B}^\pm(\lambda)$ and $\mathscr{C}^\pm(\lambda)$ have coefficients $b^\pm_{h\ell}$ and $c^\pm_{h\ell}$ respectively, determined by the identities

$$\gamma_+\big(B_h w_\ell^\pm\big) = r^{\lambda-\rho_h^B} b^\pm_{h\ell} \quad\text{and}\quad \gamma_-\big(C_h w_\ell^\pm\big) = r^{\lambda-\rho_h^C} c^\pm_{h\ell} \quad \text{on } \Gamma, \tag{2.4}$$

for h and ℓ in $\{1,\ldots,m\}$. ■

We have, [3, Th.4.5]

Theorem 2.3 *Let λ_0 in $\mathbb{C}$. There holds:*

(i) If λ_0 is a pole of $M(\lambda)^{-1}$ then λ_0 is a pole of $\mathscr{N}(\lambda)^{-1}$; let d be the order of this pole of $\mathscr{N}(\lambda)^{-1}$. Then

$$\mathscr{X}^{\lambda_0}(\mathscr{C}) = \Big\{ \frac{1}{2i\pi}\int_{\gamma(\lambda_0)} W(\lambda)\,\mathscr{N}(\lambda)^{-1}\Psi(\lambda)\,d\lambda \quad|\quad \Psi\in\mathbb{P}_{d-1}[\lambda]\otimes\mathbb{C}^{2m}\Big\},$$

with the space of polynomials $\mathbb{P}_{d-1}[\lambda]$ of degree $< d$

(ii) If a non-integer λ_0 is a pole of $\mathscr{N}(\lambda)^{-1}$ then λ_0 is a pole of $M(\lambda)^{-1}$.

2.b Formulas for a basis of $\mathfrak{W}(\lambda)$

To make this statement efficient, it remains to exhibit a basis of $\mathfrak{W}(\lambda)$. In order to do that, we use notations and concepts from [3, §2.a]. For the elementary example when L is the scalar Laplace operator, such a basis is given simply by ζ^λ and $\overline{\zeta}^\lambda$ for any $\lambda \neq 0$, where ζ is the complex writing of the cartesian variables

$$\zeta = x_1 + ix_2 = re^{i\theta} .$$

Such a result extends to our properly elliptic ADN system L as follows. We introduce the following diagonal $N \times N$ matrices for complex numbers $\lambda \in \mathbb{C}$, $\zeta \in \mathscr{C}$, $\zeta^* \in \mathscr{C}$ and $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$:

$$\begin{aligned} Z^+(\lambda;\zeta,\zeta^*;\alpha) &= \Big(\lambda(\lambda-1)\cdots(\lambda-\tau_\ell+1)\,(\alpha\zeta+\zeta^*)^{\lambda-\tau_\ell}\,\delta_{k\ell}\Big)_{1\leq k,\ell\leq N} \\ Z^-(\lambda;\zeta,\zeta^*;\alpha) &= \Big(\lambda(\lambda-1)\cdots(\lambda-\tau_\ell+1)\,(\zeta+\alpha\zeta^*)^{\lambda-\tau_\ell}\,\delta_{k\ell}\Big)_{1\leq k,\ell\leq N} \end{aligned} \tag{2.5}$$

and the Cayley symbols of L

$$L^+(\alpha) := L\big(\alpha+1, i(\alpha-1)\big) \quad \text{and} \quad L^-(\alpha) := L\big(1+\alpha, i(1-\alpha)\big). \tag{2.6}$$

Noting that

$$\partial_1 = \partial_\zeta + \partial_{\overline{\zeta}} \quad \text{and} \quad \partial_2 = i\,(\partial_\zeta - \partial_{\overline{\zeta}})$$

where ∂_ζ , $\partial_{\overline{\zeta}}$ satisfy

$$\partial_\zeta\zeta\,,\ \partial_{\overline{\zeta}}\overline{\zeta} = 1\,, \qquad \partial_\zeta\overline{\zeta}\,,\ \partial_{\overline{\zeta}}\zeta = 0\,,$$

we obtain

$$L(\partial_1,\partial_2)\,Z^\pm(\lambda;\zeta,\overline{\zeta};\alpha) = \breve{Z}^\pm(\lambda;\zeta,\overline{\zeta};\alpha)\,L^\pm(\alpha)\,, \tag{2.7}$$

where $\breve{Z}^\pm$ have a similar expression as $Z^\pm$ with τ_ℓ replaced with σ_j . According to [3, Th.2.1], the elements of $\mathfrak{W}^\pm(\lambda)$ can be obtained as Cauchy integrals in the variable α :

Lemma 2.4 *For any non-integer* $\lambda \in \mathbb{C} \setminus \mathbb{N}$ *there holds*

$$\mathfrak{W}^\pm(\lambda) = \Big\{ \frac{1}{2i\pi} \int_{|\alpha|=1} Z^\pm(\lambda;\zeta,\overline{\zeta};\alpha)\,(L^\pm)^{-1}(\alpha)\,F(\alpha)\,d\alpha \ \big| \quad F(\alpha) \in \mathfrak{A}[\alpha]\otimes\mathbb{C}^N \Big\}. \tag{2.8}$$

Due to the proper ellipticity of L , both matrices $L^\pm$ have no roots on the unit circle (and exactly m roots inside the unit circle). In order to give a sense to the above integral, we have to make precise what means a contour integral of the type

$$\int_{|\alpha|=1} (\alpha\zeta+\overline{\zeta})^\lambda\,h(\alpha)\,d\alpha$$

for a function h meromorphic in $\mathbb{C}$ without poles on $|\alpha| = 1$. There exists therefore $\rho < 1$ such that h has no pole in $\{\alpha \in \mathbb{C} \mid \rho \le \alpha \le 1\}$. We define $(\alpha\zeta + \overline{\zeta})^\lambda$ as the product $\overline{\zeta}^\lambda(1 + \alpha\frac{\zeta}{\overline{\zeta}})^\lambda$. This is well defined and analytic in α and λ for $|\alpha| \le \rho$ and for ζ in the sector $\mathscr{C}$: We can choose the branch of $\overline{\zeta}^\lambda$ that coincides with $|\zeta|^\lambda$ on the positive real axis. Then we define

$$\int_{|\alpha|=1} (\alpha\zeta + \overline{\zeta})^\lambda \, h(\alpha)\; d\alpha := \int_{|\alpha|=\rho} \overline{\zeta}^\lambda \Big(1 + \alpha\frac{\zeta}{\overline{\zeta}}\Big)^\lambda h(\alpha)\; d\alpha\,, \tag{2.9}$$

and similarly

$$\int_{|\alpha|=1} (\zeta + \alpha\overline{\zeta})^\lambda \, h(\alpha)\; d\alpha := \int_{|\alpha|=\rho} \zeta^\lambda \Big(1 + \alpha\frac{\overline{\zeta}}{\zeta}\Big)^\lambda h(\alpha)\; d\alpha\,. \tag{2.10}$$

3 The characteristic matrices for crack tips

We give now a particular form for the matrices $\mathscr{N}(\lambda)$ in connection with the situation of a crack.

The following assumption yields simpler formulas for a basis of $\mathfrak{W}^\pm(\lambda)$ and, as will be seen later, is not restrictive.

Hypothesis 3.1 *The roots of the equations* $\det L^\pm(\alpha) = 0$ *are distinct. Let* $\alpha_1^{\pm}, \ldots, \alpha_m^{\pm}$ *be these roots contained in the unit disk* $|\alpha| < 1$.

Then we have simple expressions for bases of $\mathfrak{W}^\pm(\lambda)$:

Lemma 3.2 *Under Hypothesis* 3.1*, for any* $\ell = 1, \ldots, m$ *let* $q_\ell^{\pm} \in \mathbb{C}^N$ *be a non-zero element of the kernel of* $L^\pm(\alpha_\ell^{\pm})$. *Then the following sets are bases of* $\mathfrak{W}^\pm(\lambda)$

$$w_\ell^{\pm}(\lambda) := Z^\pm(\lambda; \zeta, \overline{\zeta}; \alpha_\ell^{\pm})\, q_\ell^{\pm}, \quad \textit{for} \quad \ell = 1, \ldots, m. \tag{3.1}$$

According to (2.9) and (2.10), in the formulas giving $Z^\pm$ we simply use the expressions

$$(\alpha_\ell^{+}\zeta + \overline{\zeta})^\lambda := \overline{\zeta}^\lambda \Big(1 + \alpha_\ell^{+}\frac{\zeta}{\overline{\zeta}}\Big)^\lambda \quad \text{and} \quad (\zeta + \alpha_\ell^{-}\overline{\zeta})^\lambda := \zeta^\lambda \Big(1 + \alpha_\ell^{-}\frac{\overline{\zeta}}{\zeta}\Big)^\lambda. \tag{3.2}$$

We still make Hypothesis 3.1. Let $B^\pm$ and $C^\pm$ be the Cayley symbols of B and C, *cf* (2.6). Then there holds

$$\left\{ \begin{array}{rcl} B(\partial_1, \partial_2)\, Z^\pm(\lambda; \zeta, \overline{\zeta}; \alpha) &=& \breve{Z}_B^{\pm}(\lambda; \zeta, \overline{\zeta}; \alpha)\, B^\pm(\alpha), \\ C(\partial_1, \partial_2)\, Z^\pm(\lambda; \zeta, \overline{\zeta}; \alpha) &=& \breve{Z}_C^{\pm}(\lambda; \zeta, \overline{\zeta}; \alpha)\, C^\pm(\alpha)\,, \end{array} \right. \tag{3.3}$$

where $\breve{Z}_B^{\pm}$ and $\breve{Z}_C^{\pm}$ have similar expressions as $Z^{\pm}$ with τ_ℓ replaced with ρ_h^B and ρ_h^C respectively.

Thus using (3.2)-(3.3) we obtain that

$$B_h w_\ell^+ = \lambda(\lambda-1)\cdots(\lambda-\rho_h^B+1)\,\overline{\zeta}^{\lambda-\rho_h^B}\Big(1+\alpha_\ell^+\frac{\zeta}{\overline{\zeta}}\Big)^{\lambda-\rho_h^B} B_h^+(\alpha_\ell^+)\,q_\ell^+$$

and similarly

$$B_h w_\ell^- = \lambda(\lambda-1)\cdots(\lambda-\rho_h^B+1)\,\zeta^{\lambda-\rho_h^B}\Big(1+\alpha_\ell^-\frac{\overline{\zeta}}{\zeta}\Big)^{\lambda-\rho_h^B} B_h^-(\alpha_\ell^-)\,q_\ell^-.$$

As the trace operator γ_p satisfies $\gamma_+(\zeta)=\gamma_+(\overline{\zeta})=r$, we obtain for the coefficients $b_{h\ell}^{\pm}$ defined in (2.4)

$$b_{h\ell}^+ = \lambda(\lambda-1)\cdots(\lambda-\rho_h^B+1)\,(1+\alpha_\ell^+)^{\lambda-\rho_h^B} B_h^+(\alpha_\ell^+)\,q_\ell^+$$

and

$$b_{h\ell}^- = \lambda(\lambda-1)\cdots(\lambda-\rho_h^B+1)\,(1+\alpha_\ell^-)^{\lambda-\rho_h^B} B_h^+(\alpha_\ell^-)\,q_\ell^-.$$

Introducing the diagonal $m\times m$ matrices

$$E^B(\lambda)=\operatorname{diag}\big(\lambda(\lambda-1)\cdots(\lambda-\rho_h^B+1)\big)$$

and

$$F^{\pm}(\lambda)=\operatorname{diag}\big((1+\alpha_\ell^{\pm})^{\lambda}\big)$$

we have

$$\left\{\begin{array}{lcl}\mathscr{B}^+(\lambda) &=& E^B(\lambda)\times\mathfrak{B}^+\times F^+(\lambda)\\ \mathscr{B}^-(\lambda) &=& E^B(\lambda)\times\mathfrak{B}^-\times F^-(\lambda),\end{array}\right. \tag{3.4}$$

with $\mathfrak{B}^{\pm}$ the matrices with coefficients $\mathfrak{b}_{h\ell}^{\pm}$

$$\mathfrak{b}_{h\ell}^+ = (1+\alpha_\ell^+)^{-\rho_h^B} B_h^+(\alpha_\ell^+)\,q_\ell^+ \quad\text{and}\quad \mathfrak{b}_{h\ell}^- = (1+\alpha_\ell^-)^{-\rho_h^B} B_h^-(\alpha_\ell^-)\,q_\ell^-. \tag{3.5}$$

In a similar way, noting that for any integer n the trace $\gamma_-(\zeta^{\lambda-n})$ is equal to $r^\lambda e^{2i\pi\lambda}$ and the trace $\gamma_-(\overline{\zeta}^{\lambda-n})$ is equal to $r^\lambda e^{-2i\pi\lambda}$, while $\gamma_-\big((1+\alpha\frac{\zeta}{\overline{\zeta}})^\lambda\big)$ is equal to $(1+\alpha)^\lambda$ we obtain

$$\left\{\begin{array}{lcl}\mathscr{C}^+(\lambda) &=& e^{-2i\pi\lambda}E^C(\lambda)\times\mathfrak{C}^+\times F^+(\lambda)\\ \mathscr{C}^-(\lambda) &=& e^{2i\pi\lambda}E^C(\lambda)\times\mathfrak{C}^-\times F^-(\lambda),\end{array}\right. \tag{3.6}$$

with $\mathfrak{C}^{\pm}$ the matrices with coefficients $\mathfrak{c}_{h\ell}^{\pm}$

$$\mathfrak{c}_{h\ell}^+ = (1+\alpha_\ell^+)^{-\rho_h^C} C_h^+(\alpha_\ell^+)\,q_\ell^+ \quad\text{and}\quad \mathfrak{c}_{h\ell}^- = (1+\alpha_\ell^-)^{-\rho_h^C} C_h^-(\alpha_\ell^-)\,q_\ell^-. \tag{3.7}$$

Here the diagonal matrix $E^C(\lambda)$ is defined like $E^B(\lambda)$.

Formulas (2.3) and (3.4), (3.6) can be condensed in block form as follows

$$\mathscr{N}(\lambda) = \begin{pmatrix} E^B(\lambda) & 0 \\ 0 & E^C(\lambda) \end{pmatrix} \begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ e^{-2i\pi\lambda}\mathfrak{C}^+ & e^{2i\pi\lambda}\mathfrak{C}^- \end{pmatrix} \begin{pmatrix} F^+(\lambda) & 0 \\ 0 & F^-(\lambda) \end{pmatrix}. \tag{3.8}$$

Looking back at the definition of the diagonal matrices $E^B(\lambda)$, $E^C(\lambda)$ and $F^{\pm}(\lambda)$, we have obtained

Lemma 3.3 *The matrix $\mathscr{N}(\lambda)$ appearing in Theorem* 2.3 *has a factorization in a product of three matrices*

$$\mathscr{N}(\lambda) = E(\lambda) \begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ e^{-2i\pi\lambda}\mathfrak{C}^+ & e^{2i\pi\lambda}\mathfrak{C}^- \end{pmatrix} F(\lambda) \tag{3.9}$$

where the matrices $E(\lambda)$ and $F(\lambda)$ are diagonal. The matrix $F(\lambda)^{-1}$ is holomorphic in λ on $\mathbb{C}$ and $E(\lambda)^{-1}$ has simple poles only, and the set of its poles is

$$\big\{0, 1, \ldots, \max_{1\le h\le m} \{\rho_h^B - 1, \rho_h^C - 1\}\big\}. \tag{3.10}$$

The matrices $\mathfrak{B}^{\pm}$ and $\mathfrak{C}^{\pm}$ have coefficients $\mathfrak{b}_{h\ell}^{\pm}$ and $\mathfrak{c}_{h\ell}^{\pm}$ given by (3.5) *and* (3.7) *respectively.*

4 Poles of the inverse Mellin symbol

The aim of this section is to combine Theorem 2.3 with Lemma 3.3 in order to collect properties of the set of poles of the Mellin symbol $M(\lambda)^{-1}$ in the situation of a crack tip. We start with the following lemmas

Lemma 4.1 *The matrices $\mathfrak{B}^{\pm}$ and $\mathfrak{C}^{\pm}$ appearing in Lemma* 3.9 *are invertible.*

PROOF. It suffices to prove this for $\mathfrak{B}^{\pm}$. As the boundary system B covers the interior system L, taking $C = B$ we obtain an elliptic boundary value problem on the domain $\mathscr{C}$, and then, according to the general theory of corner domains, the associated Mellin symbol has a meromorphic inverse. Therefore, in view of Theorem 2.3 the corresponding characteristic matrix $\mathscr{N}(\lambda)$ has a meromorphic inverse. Taking $B = C$ in (3.9) we deduce that

$$\begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ e^{-2i\pi\lambda}\mathfrak{B}^+ & e^{2i\pi\lambda}\mathfrak{B}^- \end{pmatrix}$$

has a meromorphic inverse. Therefore the same holds for the matrix

$$\begin{pmatrix} \mathrm{Id} & 0 \\ -e^{-2i\pi\lambda}\mathrm{Id} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ e^{-2i\pi\lambda}\mathfrak{B}^+ & e^{2i\pi\lambda}\mathfrak{B}^- \end{pmatrix} = \begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ 0 & (e^{2i\pi\lambda} - e^{-2i\pi\lambda})\mathfrak{B}^- \end{pmatrix}$$

and the matrices $\mathfrak{B}^+$ and $\mathfrak{B}^-$ are invertible. ∎

A straightforward calculation yields:

Lemma 4.2 *Let us set* $\mathfrak{B} = (\mathfrak{B}^-)^{-1}\mathfrak{B}^+$ *and* $\mathfrak{C} = (\mathfrak{C}^-)^{-1}\mathfrak{C}^+$ *. Then there holds*

$$\mathscr{N}(\lambda) = E(\lambda) \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & e^{2i\pi\lambda}\mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ 0 & \mathfrak{C}^- \end{pmatrix} \begin{pmatrix} \big(\mathrm{Id} - e^{-4i\pi\lambda}\mathfrak{B}^{-1}\mathfrak{C}\big) & 0 \\ e^{-4i\pi\lambda}\mathfrak{C} & \mathrm{Id} \end{pmatrix} F(\lambda). \tag{4.1}$$

We deduce from this lemma a periodic structure for the set $\mathfrak{S}(M)$ of the poles of the inverse Mellin symbol M , which is called the *spectrum of the Mellin symbol*.

Theorem 4.3 *The Mellin symbol* M *associated with problem* (1.1) *satisfies the following properties: There exists an integer* $n \in \{1, \dots, m\}$ *and* n *distinct complex numbers* $\lambda_1, \dots, \lambda_n$ *with* $\operatorname{Re}\lambda_j \in [0, \frac{1}{2})$ *, such that the following inclusions hold for its spectrum* $\mathfrak{S}(M)$

$$\begin{aligned} \big\{\lambda_j + \tfrac{k}{2} \mid 1 \le j \le n,\ k \in \mathbb{Z}\big\} \setminus J \quad &\subset \quad \mathfrak{S}(M) \\ \mathfrak{S}(M) \quad &\subset \quad \big\{\lambda_j + \tfrac{k}{2} \mid 1 \le j \le n,\ k \in \mathbb{Z}\big\} \cup J \end{aligned} \tag{4.2}$$

where J *is the set of integers* (3.10).

PROOF. 1. If Hypothesis 3.1 is satisfied, we deduce from Lemma 4.2 that the set of the poles of $\mathscr{N}(\lambda)^{-1}$ is the union of the set J in (3.10) (which is the set of poles of $E(\lambda)^{-1}$) and of the set $\widetilde{J}$ of all $\lambda \in \mathbb{C}$ such that $\big(\mathrm{Id} - e^{-4i\pi\lambda}\mathfrak{B}^{-1}\mathfrak{C}\big)$ is not invertible. Therefore $\widetilde{J}$ is the set of all $\lambda \in \mathbb{C}$ such that $e^{4i\pi\lambda}$ is an eigenvalue of $\mathfrak{B}^{-1}\mathfrak{C}$. Let $s_1, \dots, s_n$ be the (distinct) eigenvalues of $\mathfrak{B}^{-1}\mathfrak{C}$. Then using Theorem 2.3, we easily obtain inclusions (4.2) by setting

$$\lambda_j \in \mathbb{C} \quad \text{with} \quad \operatorname{Re}\lambda_j \in [0, \tfrac{1}{2}), \qquad e^{4i\pi\lambda_j} = s_j. \tag{4.3}$$

2. In the case when Hypothesis 3.1 is not satisfied, it is still possible to define a smooth perturbation $[0, 1] \ni \varepsilon \mapsto L_\varepsilon$ of L, such that L_ε is still elliptic, B and C cover it, and such that the roots $\alpha^\pm_\ell(\varepsilon)$ associated with L_ε are distinct. The corresponding Mellin symbols $M_\varepsilon(\lambda)$ also smoothly depend on ε . Therefore inclusions (4.2) hold for $\mathfrak{S}(M_\varepsilon)$, thus, in the limit, for $\mathfrak{S}(M)$. ■

Remark 4.4 When the two boundary systems B and C are distinct, we have a *mixed* boundary value problem and the situation where the domain $\mathscr{C}$ is a half-space is also of interest in this case: the boundary of the half-space $\{(x_1, x_2)\ ,\ x_1 \in \mathbb{R},\ x_2 > 0\}$ is split into the two parts $\Gamma_\pm = \{(x_1, x_2)\ ,\ x_1 \in \mathbb{R}_\pm,\ x_2 = 0\}$ and the boundary conditions are given by B on Γ_+ and by C on Γ_- . Then we have a factorization of the characteristic matrix $\mathscr{N}(\lambda)$ similar to (3.8) in the form

$$\mathscr{N}(\lambda) = \begin{pmatrix} E^B(\lambda) & 0 \\ 0 & E^C(\lambda) \end{pmatrix} \begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ e^{-i\pi\lambda}\mathfrak{C}^+ & e^{i\pi\lambda}\mathfrak{C}^- \end{pmatrix} \begin{pmatrix} F^+(\lambda) & 0 \\ 0 & F^-(\lambda) \end{pmatrix},$$

with the same matrices $\mathfrak{B}^\pm$, $\mathfrak{C}^\pm$ and $F^\pm$ as in (3.8) and new matrices E^B defined as

$$E^B(\lambda) = \operatorname{diag}\big((-1)^{\rho_h^B}\lambda(\lambda-1)\cdots(\lambda-\rho_h^B+1)\big)$$

and similarly for E^C . Therefore, in that situation, we obtain that the singular exponents are distributed with a period 1 instead of $\frac{1}{2}$ according to

$$\begin{aligned} \big\{\lambda_j + k \mid 1 \le j \le n,\ k \in \mathbb{Z}\big\} \setminus J \quad \subset \quad \mathfrak{S}(M) \\ \mathfrak{S}(M) \quad \subset \quad \big\{\lambda_j + k \mid 1 \le j \le n,\ k \in \mathbb{Z}\big\} \cup J \end{aligned} \tag{4.4}$$

The fact that the singular exponents are distributed with a period 1 can be proved by a more straightforward argument than the previous analysis: it suffices to remark that if u is a singular function belonging to $\mathscr{X}^\lambda(\mathscr{C})$, then $\partial_1 u$ is a singular function belonging to $\mathscr{X}^{\lambda-1}(\mathscr{C})$. ■

5 Same boundary conditions on both sides of the crack

We consider in this section the case when the two systems of covering boundary conditions B and C are equal. As a corollary of Theorem 4.3 we obtain:

Theorem 5.1 *If $B = C$ the spectrum $\mathfrak{S}(M)$ of the Mellin symbol M associated with problem* (1.1) *satisfies the following inclusions:*

$$\big\{\ \tfrac{k}{2}\ \mid\ k \in \mathbb{Z}\big\} \setminus J \quad \subset \quad \mathfrak{S}(M) \quad \subset \quad \big\{\ \tfrac{k}{2}\ \mid\ k \in \mathbb{Z}\big\}. \tag{5.1}$$

PROOF. If Hypothesis 3.1 holds, (5.1) is a consequence of (4.2) since the operator $\mathfrak{B}^{-1}\mathfrak{C}$ is now the identity, and thus the eigenvalues s_j are all equal to 1 . Thus $n = 1$ and $\lambda_1 = 0$. If Hypothesis 3.1 does not hold, we obtain the result by a perturbation argument as before. ■

For any non-integer singularity exponent λ , the corresponding singularity space $\mathscr{X}^\lambda(\mathscr{C})$ can be described precisely.

Theorem 5.2 *If $B = C$ the singularity spaces associated with problem* (1.1) *satisfy for any non-integer exponent $\lambda = \frac{k}{2} \notin \mathbb{N}$:*

$$\mathscr{X}^\lambda(\mathscr{C}) = \big\{u = \big(r^{\lambda-\tau_1}\,\varphi_1(\theta), \ldots, r^{\lambda-\tau_N}\,\varphi_N(\theta)\big) \mid\ (\varphi_1, \ldots, \varphi_N) \in \Phi^\lambda\big\}, \tag{5.2}$$

with a m -dimensional space Φ^λ .

PROOF. We assume that Hypothesis 3.1 holds. Let λ_0 belong to $\mathfrak{S}(M) \setminus \mathbb{N}$. Since the operator $\mathfrak{B}^{-1}\mathfrak{C}$ is equal to the identity in $m \times m$ matrices, the order of λ_0 as pole of $\mathscr{N}(\lambda)$ is 1 and there holds

$$\mathscr{X}^{\lambda_0}(\mathscr{C}) = \Big\{ \int_{\gamma(\lambda_0)} \big(\widetilde{W}^+(\lambda) - e^{-4i\pi\lambda}\widetilde{W}^-(\lambda)\mathfrak{B}\big)\,(1 - e^{-4i\pi\lambda})^{-1}\,E(\lambda)^{-1}\Psi\,d\lambda, \quad \Psi \in \mathbb{C}^{2m}\Big\},$$

where $\widetilde{W}^{\pm}(\lambda) = W^{\pm}(\lambda)\big(F^{\pm}(\lambda)\big)^{-1}$. Therefore

$$\mathscr{X}^{\lambda_0}(\mathscr{C}) = \mathrm{Range}\,\big(\widetilde{W}^{+}(\lambda_0) - \widetilde{W}^{-}(\lambda_0)\mathfrak{B}\big) \times E(\lambda_0)^{-1}.$$

As λ_0 does not belong to J, $E(\lambda_0)$ is invertible. Thus

$$\mathscr{X}^{\lambda_0}(\mathscr{C}) = \mathrm{Range}\,\big(\widetilde{W}^{+}(\lambda_0) - \widetilde{W}^{-}(\lambda_0)\mathfrak{B}\big). \tag{5.3}$$

With Lemma 3.2 we now easily obtain (5.2). If Hypothesis 3.1 does not hold, we still obtain the result by perturbation. ■

Although they belong to the spectrum of the Mellin symbol, integers generally do not produce singularities in ordinary Sobolev spaces if the boundary data satisfy $g_+ = g_-$. According to the principles of [5], see also [4], for any integer $\lambda \in \mathbb{N}$ we introduce

$$\mathscr{Y}^{\lambda}(\mathscr{C}) = \big\{u \in S^{\lambda}(\mathscr{C}) \text{ solution of (1.1) with polynomial } f,\ g_+ \text{ and } g_- = g_+\big\}.$$

And the singularity space associated with such a λ is a complement $\mathscr{Z}^{\lambda}(\mathscr{C})$ in $\mathscr{Y}^{\lambda}(\mathscr{C})$ of the homogeneous polynomials of multi-degree $\lambda - \tau_k$.

Theorem 5.3 *If $B = C$ there holds for integer $\lambda \in \mathbb{N}$ large enough:*

$$\dim \mathscr{X}^{\lambda}(\mathscr{C}) = m, \quad \mathscr{X}^{\lambda}(\mathscr{C}) \textit{ is a space of polynomials and } \quad \mathscr{Z}^{\lambda}(\mathscr{C}) = \{0\}. \tag{5.4}$$

PROOF. We assume that Hypothesis 3.1 holds. Let λ belong to $\mathbb{N} \setminus J$. Then we have (5.3) and for λ large enough all columns of the matrix $W(\lambda)$ are independent polynomials. Therefore we have the first two assertions of (5.4).

If λ is an integer larger than all τ_k, σ_j and ρ_h^B, then the operator

$$(L, \gamma_+ B) : \bigotimes_{k=1}^{N} \mathbb{P}_{\lambda-\tau_k} \longrightarrow \Big(\bigotimes_{j=1}^{N} \mathbb{P}_{\lambda-\sigma_j}\Big) \otimes \mathbb{C}^m$$

has $\mathscr{X}^{\lambda}(\mathscr{C})$ as kernel and a simple evaluation of dimensions yields that it is onto. Then we deduce that $\mathscr{Y}^{\lambda}(\mathscr{C})$ coincides with $\otimes_{k=1}^{N}\mathbb{P}_{\lambda-\tau_k}$ and that $\mathscr{Z}^{\lambda}$ is reduced to zero. If Hypothesis 3.1 does not hold, we obtain the result once more by perturbation. ■

6 Mixed Dirichlet-Neumann problems for strongly coercive systems

In the former sections, we have proved that in the most general case the singularity exponents have the form $\lambda_j + \frac{k}{2}$ with $\mathrm{Re}\,\lambda_j \in [0, \frac{1}{2})$ and that when the boundary conditions B and C coincide, the λ_j are equal to 0 and that the integers in the family $\lambda_j + \frac{k}{2}$ do not produce singularities in general. We are going to investigate an opposite situation where the λ_j all satisfy $\mathrm{Re}\,\lambda_j = \frac{1}{4}$ (thus for any integer k, $\lambda_j + \frac{k}{2}$ is not integer and is an "active" singularity exponent).

Our interior operator is still an elliptic system $L = (L_{jk})$, but associated with a variational formulation. The assumption about the orders of the operator coefficients L_{jk} is that there exist $(m_1, \ldots, m_N)$ such that

$$\deg L_{jk} = m_j + m_k.$$

We moreover assume that there are given coefficients $a_{jk}^{\beta\gamma}$ defining a differential hermitian product

$$a(u,v) \quad := \quad \sum_{jk} \sum_{|\beta|=m_j,\, |\gamma|=m_k} a_{jk}^{\beta\gamma}\, \partial^\beta u_j\, \partial^\gamma \overline{v_k},$$

such that for any bounded domain Ω there holds

$$\forall u \in \mathscr{C}^\infty(\Omega)^N, \;\; \forall v \in \mathscr{C}_0^\infty(\Omega)^N, \qquad \int_\Omega \langle Lu, v\rangle \,\mathrm{d}x = \int_\Omega a(u,v)\,\mathrm{d}x\,.$$

Here $\langle f, g\rangle$ denotes the hermitian product $f \cdot \overline{g}$. Our assumption is the following

Hypothesis 6.1 *1. The form a is strongly coercive in the sense that there exists a constant $c > 0$ such that for all $\psi_j^\beta \in \mathbb{C}$, $j = 1\ldots, N$, $|\beta| = m_j$, there holds the estimate*

$$\mathrm{Re} \sum_{jk} \sum_{|\beta|=m_j,\, |\gamma|=m_k} a_{jk}^{\beta\gamma}\, \psi_j^\beta\, \overline{\psi}_k^\gamma \quad \geq \quad c \sum_j \sum_{|\beta|=m_j} |\psi_j^\beta|^2. \tag{6.1}$$

2. The system of boundary operators B is the canonical Dirichlet system of order $(m_1 - 1, \ldots, m_N - 1)$, i.e.

$$Bu = \big(u_1, \partial_n u_1, \ldots, \partial_n^{m_1-1} u_1, \ldots, u_N, \partial_n u_N, \ldots, \partial_n^{m_N-1} u_N\big)$$

and the system C is the unique system (cf [13]) *such that there holds*[(a)]

$$\begin{gathered}\forall u \in \textstyle\prod_j H^{2m_j}(\Omega), \qquad \forall v \in \prod_k H^{m_k}(\Omega), \\ \int_\Omega \langle Lu, v\rangle\,\mathrm{d}x = \int_\Omega a(u,v)\,\mathrm{d}x + \int_{\partial\Omega} \langle Cu, Bv\rangle\,\mathrm{d}\sigma.\end{gathered} \tag{6.2}$$

Let $\underline{m}$ be $\max_j m_j$. Then setting

$$\sigma_j = \underline{m} + m_j \quad \text{and} \quad \tau_k = \underline{m} - m_k,$$

we obtain that the system L is in the framework we used above, and its Mellin symbol is defined with (1.5) as usual. Of course, $m = m_1 + \cdots + m_N$ and we can check that, moreover

$$\max_h \rho_h^B = \underline{m} - 1 \quad \text{and} \quad \max_h \rho_h^C = 2\underline{m} - 1.$$

Thus the set J in (3.10) is simply

(a) Equation (6.2) is valid in any case if $\partial\Omega$ is smooth. If $\partial\Omega$ is piecewise smooth and if one of the m_j is ≥ 2, it may happen that one has to add corner contributions to the right hand side of (6.2).

$$\{0,1,\ldots,2\underline{m}-2\}.$$

The aim of this section is the proof of (compare with Theorem 4.3)

Theorem 6.2 *Under Hypotheses* 6.1*, the Mellin symbol* M *associated with problem* (1.1) *satisfies the following properties: There exists an integer* $n\in\{1,\ldots,m\}$ *and* n *distinct complex numbers* $\lambda_1^{2\pi},\ldots,\lambda_n^{2\pi}$ *with* $\operatorname{Re}\lambda_j^{2\pi}=\frac14$*, such that inclusions* (4.2) *hold for its spectrum* $\mathfrak{S}(M)$*.*

The proof of this theorem requires the consideration of problem (1.1) on any plane sector $\mathscr{C}^\omega$ of opening ω, with $\omega\in(0,2\pi]$, that is

$$\left\{\begin{array}{rcll} Lu &=& f & \text{in}\,\mathscr{C}^\omega,\\ \gamma_+(Bu) &=& g_+ & \text{on}\,\Gamma_+,\\ \gamma_-(Cu) &=& g_- & \text{on}\,\Gamma_-,\end{array}\right. \tag{6.3}$$

where Γ_+ is the ray $\theta=0$ and Γ_- the ray $\theta=\omega$.

Our first step is to deduce Theorem 6.2 from the following statement relating to the mixed problem on the half-space $\mathscr{C}^\pi$:

Theorem 6.3 *Under Hypotheses* 6.1*, the Mellin symbol* M^π *associated with problem* (6.3) *on the half-space satisfies the following properties. There exists an integer* $n\in\{1,\ldots,m\}$ *and* n *distinct complex numbers* $\lambda_1^\pi,\ldots,\lambda_n^\pi$ *with* $\operatorname{Re}\lambda_j^\pi=\frac12$*, such that the following inclusions hold for its spectrum* $\mathfrak{S}(M^\pi)$

$$\begin{array}{rcl}\{\lambda_j^\pi+k\mid 1\le j\le n,\ k\in\mathbb{Z}\}\setminus J &\subset& \mathfrak{S}(M^\pi)\\ \mathfrak{S}(M^\pi) &\subset& \{\lambda_j^\pi+k\mid 1\le j\le n,\ k\in\mathbb{Z}\}\cup J\end{array} \tag{6.4}$$

with $J=\{0,1,\ldots,2\underline{m}-2\}$*.*

PROOF OF "THEOREM 6.3 $\Rightarrow$ THEOREM 6.2". The characteristic matrix $\mathscr{N}^\pi(\lambda)$ of problem (6.3) on the half-space $\mathscr{C}^\pi$ can be factorized on a very similar way as in the case of the domain $\mathscr{C}^{2\pi}$: instead of the factorization (3.9) we find

$$\mathscr{N}^\pi(\lambda)=E^\pi(\lambda)\begin{pmatrix}\mathfrak{B}^+ & \mathfrak{B}^-\\ e^{-i\pi\lambda}\mathfrak{C}^+ & e^{i\pi\lambda}\mathfrak{C}^-\end{pmatrix}F(\lambda), \tag{6.5}$$

with the same matrices $\mathfrak{B}^\pm$, $\mathfrak{C}^\pm$ and $F(\lambda)$ as in (3.9), and

$$E^\pi(\lambda)=\begin{pmatrix}E^B(\lambda) & 0\\ 0 & E^C_\pi(\lambda)\end{pmatrix},$$

with E^B as in (3.8) and

$$E^C_\pi(\lambda)=\operatorname{diag}\big((-1)^{\rho_h^C}\,\lambda(\lambda-1)\cdots(\lambda-\rho_h^C+1)\big).$$

Therefore (6.4) holds with certain λ_j^π such that $\mathrm{Re}\,\lambda_j^\pi \in [0,1)$. Moreover, comparing the factorizations (3.9) and (6.5) we clearly have

$$\lambda_j^{2\pi} = \frac{\lambda_j^\pi}{2}\,. \tag{6.6}$$

■

It remains to prove that $\mathrm{Re}\,\lambda_j^\pi = \frac{1}{2}$. In order to do this we are going to prove the following lemma which is inspired by [11].

Lemma 6.4 *Let* $\omega \neq \pi$ *and* $\neq 2\pi$. *Then any singular exponent* λ *associated with problem* (6.3) *satisfies* $\mathrm{Re}\,\lambda \neq \underline{m} - \frac{1}{2}$.

PROOF. Let us assume that there exists λ with $\mathrm{Re}\,\lambda = \underline{m} - \frac{1}{2}$ and such that there exists a non-zero element of $S^\lambda(\mathscr{C}^\omega)$ solution of (6.3) with $f = 0$ and $g_\pm = 0$. A standard argument yields that u can be taken in $S_0^\lambda(\mathscr{C}^\omega)$, i.e. u is homogeneous. For fixed real numbers ε and ρ such that $0 < \varepsilon < \rho$, let us set

$$\mathscr{C}^\omega_{\varepsilon,\rho} := \big\{x \mid \quad \varepsilon < r < \rho \quad \text{and} \quad 0 < \theta < \omega\big\}.$$

Then the boundary of $\mathscr{C}^\omega_{\varepsilon,\rho}$ has four pieces
- its straight sides $\Gamma^0_{\varepsilon,\rho}$ and $\Gamma^\omega_{\varepsilon,\rho}$,
- its circular sides G_ε and G_ρ.

Let us set $v := \partial_1 u$. Then formula (6.2) yields

$$\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} \langle Lu, v\rangle \,\mathrm{d}x = \int_{\mathscr{C}^\omega_{\varepsilon,\rho}} a(u,v)\,\mathrm{d}x + \int_{\Gamma^0_{\varepsilon,\rho}\cup\Gamma^\omega_{\varepsilon,\rho}\cup G_\varepsilon\cup G_\rho} \langle Cu, Bv\rangle \,\mathrm{d}\sigma.$$

The degrees of homogeneity of u yield that the product $\langle Cu, Bv\rangle$ is *an homogeneous function of degree* -1. Therefore[(b)]

$$\int_{G_\varepsilon\cup G_\rho} \langle Cu, Bv\rangle \,\mathrm{d}\sigma = 0\,.$$

As u satisfies the Neumann conditions $Cu = 0$ on Γ_-, thus on $\Gamma^\omega_{\varepsilon,\rho}$, we have

$$\int_{\Gamma^\omega_{\varepsilon,\rho}} \langle Cu, Bv\rangle \,\mathrm{d}\sigma = 0\,.$$

As u satisfies the Dirichlet conditions on $\Gamma^0_{\varepsilon,\rho}$, and ∂_1 is a *tangential derivative* along $\Gamma^0_{\varepsilon,\rho}$, $v = \partial_1 u$ also satisfies the Dirichlet condition and

$$\int_{\Gamma^0_{\varepsilon,\rho}} \langle Cu, Bv\rangle \,\mathrm{d}\sigma = 0\,.$$

(b) If one has corner contributions to formula (6.2), they have the degree of homogeneity 0 and they cancel out at the two extremities of $\Gamma^0_{\varepsilon,\rho}$ and $\Gamma^\omega_{\varepsilon,\rho}$.

As $Lu = 0$, we are left with the equality

$$\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} a(u,v)\,\mathrm{d}x = 0\,.$$

But $a(u,v) = a(u,\partial_1 u) = \overline{a(\partial_1 u,u)}$. Moreover $\partial_1 a(u,u) = a(u,\partial_1 u) + a(\partial_1 u,u)$. Therefore we find that

$$\mathrm{Re}\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} \partial_1 a(u,u)\,\mathrm{d}x = 0\,.$$

Integrating by parts once more we obtain

$$\mathrm{Re}\int_{\Gamma^0_{\varepsilon,\rho}\cup\Gamma^\omega_{\varepsilon,\rho}\cup G_\varepsilon\cup G_\rho} n_1\, a(u,u)\,\mathrm{d}\sigma = 0,$$

where n_1 is the horizontal component of the outer unit normal n. But like above the homogeneity of the product $a(u,u)$ is -1, thus

$$\mathrm{Re}\int_{G_\varepsilon\cup G_\rho} n_1\, a(u,u)\,\mathrm{d}\sigma = 0.$$

As $n_1 = 0$ on $\Gamma^0_{\varepsilon,\rho}$, we are left with

$$\mathrm{Re}\int_{\Gamma^\omega_{\varepsilon,\rho}} n_1\, a(u,u)\,\mathrm{d}\sigma = 0.$$

Since we have assumed that $\omega \neq \pi$ and $\neq 2\pi$, n_1 is a non-zero constant on $\Gamma^\omega_{\varepsilon,\rho}$. Moreover estimate (6.1) yields that $\mathrm{Re}\, a(u,u)$ is ≥ 0 everywhere. Thus we obtain

$$\mathrm{Re}\ a(u,u) = 0 \quad \text{on} \quad \Gamma^\omega_{\varepsilon,\rho}.$$

Applying (6.1) once more yields

$$\forall j = 1,\dots,N, \quad \forall\beta,\ |\beta| = m_j, \qquad \partial^\beta u_j = 0 \quad \text{on} \quad \Gamma^\omega_{\varepsilon,\rho}.$$

Therefore for all $\ell = 0,\dots,m_j - 1$ the restriction of $\partial^\ell_n u_j$ to $\Gamma^\omega_{\varepsilon,\rho}$ is a polynomial p^ℓ_j. As u_j is homogeneous with non-integer degree, this polynomials p^ℓ_j are necessarily 0. Therefore u satisfies the Dirichlet conditions $Bu = 0$ on $\Gamma^\omega_{\varepsilon,\rho}$. As u also satisfies the Neumann conditions on $\Gamma^\omega_{\varepsilon,\rho}$, finally u satisfies the *Cauchy conditions* on $\Gamma^\omega_{\varepsilon,\rho}$. As $Lu = 0$, by the Cauchy uniqueness theorem, $u \equiv 0$. ■

Lemma 6.5 *Any singular exponent* λ *associated with problem* (6.3) *satisfies* $\mathrm{Re}\,\lambda \neq \underline{m} - 1$.

PROOF. This result is classical and can easily be proved like before by considering $u \in S^\lambda_0(\mathscr{C}^\omega)$ solution of problem (6.3) with $f = 0$, $g_\pm = 0$ and $\mathrm{Re}\,\lambda = \underline{m} - 1$. Then it suffices to apply formula (6.2) with u and $v = u$. Then we find that

$$\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} a(u,u)\,\mathrm{d}x = 0,$$

whence we deduce with (6.1) that $\forall j = 1,\ldots,N$, $\forall\beta$, $|\beta| = m_j$, $\partial^\beta u_j = 0$. Therefore u_j is a polynomial of degree $< m_j$. As all its derivatives of order $< m_j$ are zero on $\Gamma^0_{\varepsilon,\rho}$, we deduce that $u_j = 0$, whence $u \equiv 0$. ∎

For each $\omega \in (0, 2\pi]$, let $\xi(\omega)$ be defined as

$$\xi(\omega) = \min\bigl\{\operatorname{Re}\lambda - \underline{m} + 1 \mid \quad \operatorname{Re}\lambda \geq \underline{m} - 1 \quad \text{and} \quad \lambda \in \mathfrak{S}(M^\omega)\bigr\}.$$

The ellipticity of problem (6.3) implies that in any strip $\operatorname{Re}\lambda \in [a,b]$ there is at most a finite number of elements of $\mathfrak{S}(M^\omega)$. Thus $\xi(\omega)$ is always > 0. We have

Lemma 6.6 *As ω tends to 0, $\xi(\omega)$ tends to $+\infty$.*

PROOF. We fix ε, ε', ρ' and ρ with $0 < \varepsilon < \varepsilon' < \rho' < \rho$ and take a smooth function $\chi \geq 0$ on $\mathbb{R}$ which is $\equiv 1$ for $r \in [\varepsilon', \rho']$ and $\equiv 0$ for $r \leq \varepsilon$ and $r \geq \rho$. Let $\lambda \in \mathfrak{S}(M^\omega)$ with $\operatorname{Re}\lambda > \underline{m} - 1$ and u be a corresponding singular function in $S^\lambda_0(\mathscr{C}^\omega)$. We apply formula (6.2) for u and $v = \chi u$ on $\Omega = \mathscr{C}^\omega_{\varepsilon,\rho}$. We find that

$$\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} a(u,\chi u)\,\mathrm{d}x = 0.$$

There exists a sesqui-linear form b of order m and $m-1$ on its first and second argument such that

$$a(u,\chi u) = \chi a(u,u) + b(u,u).$$

As $u_j = r^{\lambda-\tau_j}\varphi_j(\theta)$, setting $\varphi = (\varphi_1,\ldots,\varphi_N)$ and denoting $\mathbf{m} = (m_1,\ldots,m_N)$ and $\mathbf{H^m}$ the product of Sobolev spaces $\prod_j H^{m_j}$, we obtain the estimates

$$\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} \chi a(u,u)\,\mathrm{d}x \;\;\geq\;\; 2\beta(\operatorname{Re}\lambda)\,|\varphi|^2_{\mathbf{H^m}(0,\omega)} \tag{6.7}$$

and

$$\Bigl|\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} b(u,u)\,\mathrm{d}x\Bigr| \;\;\leq\;\; \gamma(\operatorname{Re}\lambda)\,\|\varphi\|_{\mathbf{H^m}(0,\omega)}\,\|\varphi\|_{\mathbf{H^{m-1}}(0,\omega)},$$

with positive “constants” $\beta(\operatorname{Re}\lambda)$ and $\gamma(\operatorname{Re}\lambda)$ which depend continuously on $\operatorname{Re}\lambda$ and not on ω. As φ satisfies the Dirichlet conditions on $\theta = 0$, we have the Poincaré estimate in the form

$$\|\varphi\|_{\mathbf{H^{m-1}}(0,\omega)} \;\leq\; c\,\omega\,|\varphi|_{\mathbf{H^m}(0,\omega)}.$$

Combining the five previous formulas we obtain that

$$\frac{\beta(\operatorname{Re}\lambda)}{c\,\omega} \;\;\leq\;\; \gamma(\operatorname{Re}\lambda),$$

in other words that there exist a *continuous function* δ of $\operatorname{Re}\lambda$ such that there holds

$$\frac{1}{\omega} \;\leq\; \delta(\operatorname{Re}\lambda), \qquad \forall\lambda \in \mathfrak{S}(M^\omega).$$

Therefore $\operatorname{Re}\lambda \to +\infty$ as $\omega \to 0$. ∎

As problem (6.3) is symmetric, there holds

Lemma 6.7 *If λ belongs to $\mathfrak{S}(M^\omega)$, then $2(\underline{m}-1)-\overline{\lambda}$ belongs to $\mathfrak{S}(M^\omega)$ too.*

End of Proof of Theorem 6.3. We already know, *cf* Remark 4.4, that we have the inclusions (6.4) with complex generators λ_j^π such that $\operatorname{Re}\lambda_j^\pi \in [0,1)$. Let us recall that according to the factorization (6.5) these λ_j^π satisfy $e^{2i\pi\lambda_j} = s_j$ with s_j the eigenvalues of $\mathfrak{B}^{-1}\mathfrak{C}$, *cf* also (4.3). As the reunion of boundary conditions B and C is a Dirichlet system of order $2m_j-1$ with respect to the j-th argument u_j, the matrix

$$\begin{pmatrix} \mathfrak{B}^+ & \mathfrak{B}^- \\ \mathfrak{C}^+ & \mathfrak{C}^- \end{pmatrix}$$

can be seen as a Wronskian and is non-singular. Therefore 1 is not an eigenvalue of $\mathfrak{B}^{-1}\mathfrak{C}$ and λ_j^π is $\neq 0$.

It remains to prove that the λ_j^π satisfy

$$\forall j = 1,\ldots,n, \qquad \operatorname{Re}\lambda_j^\pi = \tfrac{1}{2}. \tag{6.8}$$

Let us assume that (6.8) does not hold. This means that there exists at least one λ_j^π such that $\operatorname{Re}\lambda_j^\pi \neq \frac{1}{2}$. Thus we have either $\operatorname{Re}\lambda_j^\pi < \frac{1}{2}$ or $\operatorname{Re}\lambda_j^\pi > \frac{1}{2}$. In the latter case, applying Lemma 6.7 we obtain that $2(\underline{m}-1)-\overline{\lambda_j^\pi}$ also belongs to $\mathfrak{S}(M^\pi)$. As $\lambda_j^\pi \neq 0$, the number $2(\underline{m}-1)-\overline{\lambda_j^\pi}$ is not integer and for any integer k, $2(\underline{m}-1)-\overline{\lambda_j^\pi}+k$ belongs to $\mathfrak{S}(M^\pi)$, in particular $1-\overline{\lambda_j^\pi}$ whose real part is $<\frac{1}{2}$.

Anyway, using once more that $\lambda_j^\pi \neq 0$ and the periodicity, we obtain the existence of a non-integer element λ_0^π of $\mathfrak{S}(M^\pi)$ such that $\operatorname{Re}\lambda_0^\pi \in [\underline{m}-1, \underline{m}-\frac{1}{2})$. By Lemma 6.5, $\operatorname{Re}\lambda_0^\pi \neq \underline{m}-1$, hence

$$\operatorname{Re}\lambda_0^\pi \in (\underline{m}-1,\ \underline{m}-\tfrac{1}{2}).$$

The elements of $\mathfrak{S}(M^\omega)$ are continuous with respect to ω and for any $\omega \in (0,\pi)$ we can choose λ_0^ω such that the application $\omega \mapsto \lambda_0^\omega$ is continuous on $(0,\pi]$. Moreover Lemma 6.5 gives that $\forall\omega \in (0,\pi]$, $\operatorname{Re}\lambda_0^\omega > \underline{m}-1$. Then Lemma 6.6 yields that $\operatorname{Re}\lambda_0^\omega$ tends to $+\infty$ as $\omega \to 0$. As $\operatorname{Re}\lambda_0^\omega$ is a continuous function, there exists $\omega \in (0,\pi)$ such that $\operatorname{Re}\lambda_0^\omega = \underline{m}-\frac{1}{2}$, which contradicts Lemma 6.4. Therefore we have proved (6.8). ∎

7 Mixed Dirichlet-Neumann problems for general elasticity

Let us recall that the equations of linear two-dimensional elasticity are based on the bilinear form a acting on displacements $u = (u_1, u_2)$ and $v = (v_1, v_2)$ as

$$a(u, v) := \sum_{ijkl} A^{ijkl} e_{ij}(u)\, e_{kl}(v),$$

where $e_{ij}(u)$ is the linearized strain tensor $\frac{1}{2}(\partial_i u_j + \partial_j u_i)$ and A^{ijkl} is the rigidity matrix of the material. The rigidity matrix satisfies the following symmetry properties

$$A^{ijkl} = A^{jikl} = A^{klij},$$

and the following positivity property: There exists a constant $c > 0$ such that for all 2×2 symmetric matrices (τ_{ij}), there holds the estimate

$$\sum_{ijkl} A^{ijkl} \tau_{ij}\, \tau_{kl} \quad \geq \quad c \sum_{ij} |\tau_{ij}|^2. \tag{7.1}$$

The operator L such that

$$\forall u \in \mathscr{C}^\infty(\Omega)^2,\ \ \forall v \in \mathscr{C}_0^\infty(\Omega)^2, \qquad \int_\Omega \langle Lu, v\rangle \,\mathrm{d}x = \int_\Omega a(u, v)\,\mathrm{d}x \tag{7.2}$$

is elliptic of multi-order 2 in the sense of [1] but does not satisfy the strong coercivity property (6.1). That is why Theorem 6.2 does not apply directly in this situation. Nevertheless, there still holds

Theorem 7.1 *With the above property* (7.1) *on* A^{ijkl}, *the Mellin symbol* M *associated with the elasticity problem* (1.1) *on the crack domain* $\mathscr{C}$ *with* L *given by* (7.2) *and with Dirichlet conditions* B *and Neumann conditions* C *satisfies the following properties: There exists an integer* $n \in \{1, 2\}$ *and* n *distinct complex numbers* $\lambda_1^{2\pi}, \dots, \lambda_n^{2\pi}$ *with* $\operatorname{Re}\lambda_j^{2\pi} = \frac{1}{4}$, *such that inclusions* (4.2) *hold for its spectrum* $\mathfrak{S}(M)$.

The proof follows the same lines as the proof of Theorem 6.2. Only differ the arguments in Lemmas 6.4 and 6.6.

Concerning Lemma 6.4, we take as function v: $\partial_{\tau_\omega} u$ where τ_ω is the tangential derivative along the side Γ_- and we start from the Green formula

$$\int_{\mathscr{C}^\omega_{\varepsilon,\rho}} \langle Lv, u\rangle\,\mathrm{d}x = \int_{\mathscr{C}^\omega_{\varepsilon,\rho}} a(u, v)\,\mathrm{d}x + \int_{\Gamma^0_{\varepsilon,\rho}\cup\Gamma^\omega_{\varepsilon,\rho}\cup G_\varepsilon\cup G_\rho} \langle Cv, Bu\rangle\,\mathrm{d}\sigma.$$

As u satisfies the Dirichlet conditions $Bu = 0$ on Γ_+, thus on $\Gamma^0_{\varepsilon,\rho}$, we have

$$\int_{\Gamma^0_{\varepsilon,\rho}} \langle Cv, Bu\rangle\,\mathrm{d}\sigma = 0\,.$$

As u satisfies the Neumann conditions on $\Gamma^{\omega}_{\varepsilon,\rho}$, and ∂_{τ_ω} is a *tangential derivative* along $\Gamma^{\omega}_{\varepsilon,\rho}$, $v = \partial_{\tau_\omega} u$ also satisfies the Neumann condition and

$$\int_{\Gamma^{\omega}_{\varepsilon,\rho}} \langle Cv, Bu \rangle \,\mathrm{d}\sigma = 0\,.$$

As $Lu = 0$, there also holds $Lv = 0$ and we are left with the equality

$$\int_{\mathscr{C}^{\omega}_{\varepsilon,\rho}} a(u,v)\,\mathrm{d}x = 0\,.$$

Like in the proof of Lemma 6.4, as $a(u,u)$ is nonnegative we obtain that

$$a(u,u) = 0 \quad \text{on} \quad \Gamma^{0}_{\varepsilon,\rho}.$$

Then inequality (7.1) yields that

$$\forall i,j = 1,2, \qquad e_{ij}(u) = 0 \quad \text{on} \quad \Gamma^{0}_{\varepsilon,\rho}.$$

As u satisfies the Dirichlet conditions on $\Gamma^{0}_{\varepsilon,\rho}$ we deduce from the above equality that $\partial_n u_1 = \partial_n u_2 = 0$ on $\Gamma^{0}_{\varepsilon,\rho}$. Finally u satisfies the *Cauchy conditions* on $\Gamma^{0}_{\varepsilon,\rho}$ and we conclude as previously.

In Lemma 6.6, everything works in the same way, except that, in order to obtain the estimate (6.7) which now takes the form

$$\int_{\mathscr{C}^{\omega}_{\varepsilon,\rho}} \chi a(u,u)\,\mathrm{d}x \;\;\geq\;\; 2\beta(\operatorname{Re}\lambda)\,|\varphi|^2_{\mathbf{H}^1(0,\omega)}$$

we have to use Korn inequality which, thanks to the Dirichlet conditions on $\Gamma^{0}_{\varepsilon,\rho}$, holds with a constant uniform with respect to ω.

Remark 7.2 If we consider a three-dimensional elasticity problem in a domain exterior to a bounded two-dimensional manifold $\mathscr{S}$ with boundary $\partial\mathscr{S}$ (we call $\mathscr{S}$ the screen region), we have to determine the singularities of generalized elasticity problems on the two-dimensional domain $\mathscr{C}^{2\pi}$: these generalized elasticity problems are obtained by freezing the tangential variable along the edge $\partial\mathscr{S}$ at each point of $\partial\mathscr{S}$. We can prove by the same techniques as above that these generalized elasticity problems satisfy the conclusions of Theorem 7.1 too. ∎

8 Consequences for the regularity

The boundary value problems that we have considered in sections 6 and 7 have a variational formulation and, if posed in a bounded domain Ω, have a unique solution in the variational space, which is the subspace of $\mathbf{H}^{\mathbf{m}}$ ($\mathbf{H}^1$ in the situation of elasticity) with Dirichlet boundary conditions. Let us assume that $\Omega = \Omega_0 \setminus \mathscr{S}$ where Ω_0 is a smooth bounded domain and $\mathscr{S}$ is a smooth segment (or arc) whose closure is contained in Ω_0. Two trace operators are associated with $\mathscr{S}$: γ_+ and γ_-, corresponding to the choice of two boundary operators B and C and the boundary value problem is now, instead (1.1)

$$\begin{cases} Lu &= f \quad \text{in } \Omega, \\ \gamma_+(Bu) &= g_+ \quad \text{on } \mathscr{S}, \\ \gamma_-(Cu) &= g_- \quad \text{on } \mathscr{S}, \end{cases} \tag{8.1}$$

Let us assume for simplicity that $g_\pm = 0$. We suppose that f belongs to $\mathbf{H}^{s-\mathbf{m}}(\Omega)$ with positive s.

If the boundary conditions B and C on $\mathscr{S}$ coincide (and are equal to Dirichlet or Neumann), then u belongs to $\mathbf{H}^{\mathbf{m}+s}(\Omega)$ if $s < \frac{1}{2}$, this regularity being generically optimal.

If B is Dirichlet and C Neumann (or the converse), then u belongs to $\mathbf{H}^{\mathbf{m}+s}(\Omega)$ if $s < \frac{1}{4}$, this regularity being generically optimal, too.

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