Singularities of electromagnetic fields in polyhedral domains

Martin COSTABEL and Monique DAUGE

Abstract. *In this paper, we investigate the singular solutions of time harmonic Maxwell equations in a domain which has edges and polyhedral corners. It is now well known that in the presence of non-convex edges, the solution fields have no square integrable gradients in general and that the main singularities are the gradients of singular functions of the Laplace operator. We show how this type of result can be derived from the classical Mellin analysis, and how this analysis leads to sharper results concerning the singular parts which belong to* H^1 *: For the generating singular functions, we exhibit simple and explicit formulas based on (generalized) Dirichlet and Neumann singularities for the Laplace operator. These formulas are more explicit than the results announced in our note [9].*

Introduction

Solutions of time-harmonic three-dimensional Maxwell equations

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\,\mu\boldsymbol{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\,\varepsilon\boldsymbol{E} = \boldsymbol{J}$$

with electric or magnetic boundary conditions have singularities near corners and edges of the boundary of the domain. This well-known fact has, for example, important consequences for the construction of numerical approximation of the solution. Just as for other elliptic boundary value problems, the singularities can be analyzed by standard techniques [15, 18, 13, 10] that go back to Kondratev's technique of Mellin transformation.

The special form of Maxwell equations allow to go further in the analysis of the regularity and the singularities. Some known results are:

- the H^1 regularity for convex domains, Saranen [22],
- the $H^{1/2}$ regularity for Lipschitz domains, Costabel [7],
- a description of singular functions for corners with a smooth basis, Saranen [21],

- corner singularities for the corresponding two-dimensional problem, MOUSSAOUI [19], see also section 3.

Further regularity results can be found in KŘIČEK - NEITTAANMÄKI [17], HAZARD - LENOIR [14], AMROUCHE - BERNARDI - DAUGE - GIRAULT [1]. In all these cases, the main singularity is the gradient of a singular function of the Dirichlet or Neumann problem for the Laplace operator, thus is reduced to a problem that can be considered as well known [13, 10]. This relation can be extended to a more general class of piecewise smooth domains that have "screen" or "crack" parts, BIRMAN - SOLOMYAK and FILONOV [3, 4, 11].

We will show that, in such a class of domains, not only the first, but *all singular functions* for the Maxwell boundary value problems can be obtained in simple ways from Dirichlet or Neumann singular functions of the Laplace operator.

More generally, we will assume that the domain Ω is polyhedral, that is, its boundary consists of plane faces, straight edges, and corner points: such a polyhedron needs not to be Lipschitz nor simply connected. Thus Ω can have screen parts, in which case there is only H^s regularity with $s < 1/2$. We find another type of non-Lipschitz domains with the same bad regularity: These domains are, as exemplified by a domain between two cones with the same vertex, not locally simply connected. Here the simple equation

"Maxwell regularity = Dirichlet regularity −1 "

is violated: The Dirichlet problem can even have H^2 regularity in such a case. The singular functions at such a corner are generated by topological objects: the elements of the cohomology space of the base of the cone, see section 7.

For a better understanding of this new phenomenon, consider the case of a domain between two circular cones with the same vertex and the same axis. In spherical coordinates we have with $0 < \theta_0 < \theta_1 < \pi$ in a ball $B(0, \rho_0)$:

$$\Omega \cap B(0,\rho_0) = \{(\rho,\theta,\varphi) \mid \quad \rho \in (0,\rho_0),\ \theta_0 < \theta < \theta_1,\ \varphi \in [0,2\pi)\}.$$

We consider the functions

$$\Phi(\rho,\theta,\varphi) = \log\tan\frac{\theta}{2} \quad \text{and} \quad \Psi(\rho,\theta,\varphi) = \varphi.$$

Both functions are harmonic, and with $r = \sqrt{x^2+y^2} = \rho\sin\theta$ we have

$$\boldsymbol{u} := \mathbf{grad}\,\Phi = \frac{1}{\rho}\Big(\frac{xz}{r^2},\ \frac{yz}{r^2},\ -1\Big)^{\mathrm{T}} \quad \text{and} \quad \boldsymbol{v} := \mathbf{grad}\,\Psi = \Big(-\frac{y}{r^2},\ \frac{x}{r^2},\ 0\Big)^{\mathrm{T}}.$$

Both $\boldsymbol{u}$ and $\boldsymbol{v}$ are harmonic vector fields:

$$\mathbf{curl}\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{v} = 0, \quad \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{v} = 0.$$

On $\partial\Omega$ near the vertex, $\boldsymbol{u}$ is normal and $\boldsymbol{v}$ is tangential. Thus $\boldsymbol{u}$ satisfies "electric" and $\boldsymbol{v}$ "magnetic" boundary conditions. Both functions are in H^s near the vertex for $s < 1/2$. They correspond to electrostatic and magnetostatic singularities, respectively.

Maxwell's equations are not an elliptic system. But the elimination of one of the two fields $\boldsymbol{E}$ or $\boldsymbol{H}$ yields a variational formulation in a special "physical" energy space which is $X := H(\mathrm{div}) \cap H(\mathbf{curl})$, the space of square integrable vector fields with square integrable curl and divergence. The underlying boundary value problem is then a second order elliptic system, but set in a non-standard space. Instead of this space X, also the Sobolev space H^1 could be used in a similar variational formulation. Both variational formulations are based on the same bilinear form and both have unique solutions which, for non-convex domains, will not coincide in general. The second one is the projection in X of the first one onto H^1. But the most important fact is that only the formulation in the space X gives back a solution of the original Maxwell equations.

This situation is important for numerical approximations: If one uses standard finite elements that are contained in H^1, then the "true" solution in X cannot be approximated, and mesh refinement at the corners and edges does not help. A possible solution is to augment the finite element space by the explicitly known singular functions, Assous - Ciarlet - Sonnendrücker [2] and Bonnet - Hazard - Lohrengel [5]. In such a method, the approximation is determined by the regularity of the regular part, i.e. the solution minus the singular function. This regularity can be quite different for different choices of the singular function. In particular, if the singular function is not constructed by our explicit formulas, but from abstract principles, then this regularity can be quite low, typically H^s with $s < 4/3$, see sections 3 and 8.

0.a Plan

The outline of our paper is as follows:

In section 1, we start from the classical Maxwell equations with general bounded permeability μ and permittivity ε and deduce equivalent variational formulations in spaces of X-type and we conclude this section by an important result about the H^1 regularity of the divergence (Theorem 1.3). In section 2, assuming that μ and ε are piecewise smooth in a polyhedral partition of the domain Ω, we obtain formulations of the problems in a boundary value form. From then on, we assume that ε and μ are smooth. In subsection 2.c, we introduce the variational formulation in H^1-type subspaces of the X-type spaces. For the analysis of singularities, we can then even freeze coefficients at the corners and restrict ourselves to the simplest case $\varepsilon = \mu = 1$. Before presenting in section 4 the general principles of elliptic regularity and singularities in three-dimensional domains with edges and corners in the X-type spaces, we devote section 3 to simpler and more explicit results for two-dimensional domains (polygons).

In section 5, we describe the two-dimensional generators of the edge singularities (Lemma 5.1). In section 6, we prepare for the Maxwell 3D corner singularities by recalling the properties of the Laplace operator in 3D cones for Dirichlet and Neumann conditions. All our explicit formulas for all the Maxwell corner singularities are constructed in section 7, and our results are summarized in subsection 7.e: we have got a classification in three main types, for example concerning the electric field,

- The gradients of Dirichlet Laplace singularities,
- The divergence free fields whose curls are gradients of Neumann Laplace singularity,
- The fields whose divergences are Laplace Dirichlet singularities.

When the domain is not locally simply connected in the neighborhood of the corner, the first two types are enriched by topological singularities of similar structure (the notion of Dirichlet and Neumann Laplace singularities has to be extended in a suitable way), but this concerns only the singularity exponents -1 and 0, respectively. The other singularity exponents are those of the Laplace Neumann problem and those of the Laplace Dirichlet problem ± 1 (Table 1). Concerning the magnetic field, the roles of Neumann and Dirichlet conditions are interchanged. Finally if we go back to the original Maxwell equations, we find out that only the first two types are present (Table 2).

We conclude our paper in section 8 by results about the main singularities in non-convex domains in the X-type spaces and in H^1-type spaces. While they are gradients of Laplace singularities in the first case, they can only be described as sorts of Stokes singularities in the second case, see KOZLOV - MAZYA - ROSSMANN [16].

0.b Domains and Sobolev spaces

We end this introduction by a few definitions about domains and spaces. We want to consider rather general piecewise smooth domains which need not to be Lipschitz, in general: such domains can easily appear in applications and some of them have already been studied as mentioned above.

As in [10], the definition of the classes of domains is recursive. Let $B(x,r)$ denote the ball of center x and radius r. In $\mathbb{R}^2$ we define:

- The *2D corner domains* as bounded domains Ω in $\mathbb{R}^2$ or $\mathbb{S}^2$ such that in each point x of the boundary there exists $r_x > 0$ small enough such that to each connected component $\Omega_{x,i}$, of $\Omega \cap B(x,r_x)$ belongs a diffeomorphism $\chi_{x,i}$ transforming $\Omega_{x,i}$ into a neighborhood of the corner 0 of a plane sector of opening in $(0,2\pi]$, x being sent into 0.
- The *pseudo-polygonal domains* as the 2D corner domains with straight sides (indeed any bounded domain whose boundary is a finite union of segments).

In $\mathbb{R}^3$ we define:

- The *3D corner domains* as bounded domains Ω in $\mathbb{R}^3$ such that in each point x of the boundary there exists $r_x > 0$ small enough such that to each connected component $\Omega_{x,i}$ of $\Omega \cap B(x,r_x)$ belongs a diffeomorphism $\chi_{x,i}$ transforming $\Omega_{x,i}$ into a neighborhood of the corner 0 of a cone $\Gamma_{x,i}$ of the form $\{x \in \mathbb{R}^3,\ x/|x| \in G_{x,i}\}$ with $G_{x,i}$ a 2D corner domain of $\mathbb{S}^2$, x being sent into 0.
- The *pseudo-polyhedral domains* as the 3D corner domains with straight faces (indeed any bounded domain whose boundary is a finite union of polygons).

We say that Ω in one of these classes is *locally simply connected* if for any x in its boundary, $\Omega \cap B(x, r_x)$ is simply connected.

The space $H^1(\Omega)$ is the space of complex-valued distributions $u \in \mathscr{D}'(\Omega)$ which belong to $L^2(\Omega)$ and such that each component of their gradients belongs to $L^2(\Omega)$. The space $H^{1/2}(\partial\Omega)$ is the space of traces of $H^1(\Omega)$ where it is understood that $\partial\Omega$ is the "unfolded" boundary of Ω, that is, in the neighborhood of each point $x \in \overline{\Omega} \setminus \Omega$ in the topological boundary of Ω, $\partial\Omega$ is the disjoint union of the parts of the boundaries of $\Omega_{x,i}$ which are contained in $\overline{\Omega} \setminus \Omega$. The space $H^{-1/2}(\partial\Omega)$ is the dual space of $H^{1/2}(\partial\Omega)$.

Moreover, we introduce the spaces $H(\mathbf{curl}\,;\Omega)$ and $H(\mathrm{div}\,;\Omega)$:

$$H(\mathbf{curl}\,;\Omega) = \{\boldsymbol{u} \in \mathscr{D}'(\Omega)^3 \mid \quad \boldsymbol{u} \in L^2(\Omega)^3,\ \mathbf{curl}\,\boldsymbol{u} \in L^2(\Omega)^3\} \tag{0.1}$$

$$H(\mathrm{div}\,;\Omega) = \{\boldsymbol{u} \in \mathscr{D}'(\Omega)^3 \mid \quad \boldsymbol{u} \in L^2(\Omega)^3,\ \mathrm{div}\,\boldsymbol{u} \in L^2(\Omega)\}. \tag{0.2}$$

More generally, for ρ a 3×3 matrix with L^∞ elements on Ω, we introduce the space $H(\mathrm{div}\,;\rho\,;\Omega)$:

$$H(\mathrm{div}\,;\rho\,;\Omega) = \{\boldsymbol{u} \in \mathscr{D}'(\Omega)^3 \mid \quad \boldsymbol{u} \in L^2(\Omega)^3,\ \mathrm{div}(\rho\boldsymbol{u}) \in L^2(\Omega)\}. \tag{0.3}$$

Note that the space $H(\mathrm{div}\,;\Omega)$ defined above is a particular case of the previous definition with $\rho = I_3$.

For any $\boldsymbol{u} \in H(\mathbf{curl}\,;\Omega)$, the tangential trace $\boldsymbol{u} \times \boldsymbol{n}$ is well defined in $H^{-1/2}(\partial\Omega)^3$ due to the Green formula:

$$\forall \boldsymbol{v} \in H^1(\Omega)^3 : \quad \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{v} = \langle \boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\Omega}. \tag{0.4}$$

Similarly, for any $\boldsymbol{u} \in H(\mathrm{div}\,;\rho\,;\Omega)$, the normal trace $(\rho\boldsymbol{u}) \cdot \boldsymbol{n}$ is well defined in $H^{-1/2}(\partial\Omega)$ by the Green formula:

$$\forall \varphi \in H^1(\Omega) : \quad \int_\Omega \mathrm{div}(\rho\boldsymbol{u})\,\varphi + \rho\boldsymbol{u} \cdot \mathbf{grad}\,\varphi = \langle (\rho\boldsymbol{u}) \cdot \boldsymbol{n}, \varphi \rangle_{\partial\Omega}. \tag{0.5}$$

1 Variational formulations of Maxwell's equations

In this section, we assume that Ω is a 3D corner domain as defined above and quite weak hypotheses about the permeability μ and the permittivity ε. The results of this section are also true for general Lipschitz domains, but since we are only interested in piecewise smooth domains later on, we do not try to look at the most general possible class of domains here.

After the formulation of the classical time harmonic Maxwell equations, we investigate variational formulations for either the electric field or the magnetic field. We prove equivalences and a regularity result for the divergences of the fields (or for a Lagrange multiplier – pseudo-pressure – in the case of a saddle-point formulation).

1.a Time harmonic Maxwell's equations

Let ε and μ two complex 3×3 matrices with L^∞ elements on Ω such that their symmetric part is positive in the sense that there exist $0 < \rho_0 \leq \rho_1$ such that for all $x \in \Omega$ and for all $\xi \in \mathbb{C}^3$:

$$\rho_0|\xi|^2 \leq \mathrm{Re}(\varepsilon(x)\xi \cdot \bar{\xi}) \leq \rho_1|\xi|^2 \quad \text{and} \quad \rho_0|\xi|^2 \leq \mathrm{Re}(\mu(x)\xi \cdot \bar{\xi}) \leq \rho_1|\xi|^2.$$

The classical time harmonic Maxwell equations at the frequency ω in a body occupying Ω, with permeability μ and permittivity ε are

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\,\mu\boldsymbol{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\,\varepsilon\boldsymbol{E} = \boldsymbol{J} \quad \text{in} \quad \Omega. \tag{1.1a}$$

Here $\boldsymbol{E}$ is the electric part and $\boldsymbol{H}$ the magnetic part of the electromagnetic field. The right hand side $\boldsymbol{J}$ is the current density. The exterior boundary conditions on $\partial\Omega$ are those of the perfect conductor ($\boldsymbol{n}$ denotes the unit outer normal on $\partial\Omega$):

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{and} \quad \mu\boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.1b}$$

If the body is formed by several different homogeneous media, ε and μ are piecewise constant and there are internal transmission conditions at the interfaces contained in the functional formulation (see §2). Equations (1.1a) hide equations on the divergence of the fields, as soon as ω is not 0: taking the divergence of (1.1a) leads to

$$\mathrm{div}(\varepsilon\boldsymbol{E}) = \frac{1}{i\omega}\,\mathrm{div}\,\boldsymbol{J} \quad \text{and} \quad \mathrm{div}(\mu\boldsymbol{H}) = 0. \tag{1.1c}$$

In order to prepare for the Mellin approach of singularities, we prefer an elliptic formulation of the equations, which can be clearly explained in this framework of very general ε and μ. We obtain such a formulation by the elimination of one of the fields (either $\boldsymbol{H}$ or $\boldsymbol{E}$). But, before writing a system of partial differential equations with boundary conditions (see §2), we have to describe *variational formulations*, the reverse order of steps leading almost surely to wrong formulations.

Let us assume that $\boldsymbol{J} \in H(\mathrm{div}\,;\Omega)$, i.e. $\boldsymbol{J}$ belongs to $L^2(\Omega)^3$ and its divergence $\mathrm{div}\,\boldsymbol{J}$ belongs to $L^2(\Omega)$. The equations (1.1a) and (1.1c) yield immediately that if $\boldsymbol{E}$ and $\boldsymbol{H}$ are in $L^2(\Omega)^3$, then they belong respectively to the following spaces

$$\boldsymbol{E} \in H(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\varepsilon\,;\Omega) \qquad \text{and} \qquad \boldsymbol{H} \in H(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\mu\,;\Omega).$$

Taking account of the boundary conditions (1.1b), we obtain that

$$\boldsymbol{E} \in X_N \qquad \text{and} \qquad \boldsymbol{H} \in X_T,$$

with

$$X_N = \{\boldsymbol{u} \in H(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\varepsilon\,;\Omega) \mid \quad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\}$$

and

$$X_T = \{\boldsymbol{u} \in H(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\mu\,;\Omega) \mid \quad (\mu\boldsymbol{u}) \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\}.$$

These spaces are our variational spaces.

1.b Variational formulation for the electric field

Thus $\boldsymbol{E} \in X_N$. Let $\widetilde{\boldsymbol{E}} \in X_N$. As a consequence of the assumptions, μ is invertible. Let us integrate the first equation of (1.1a) versus ${}^{\mathrm{T}}\mu^{-1}\widetilde{\boldsymbol{E}}$, and the second versus $i\omega\,\widetilde{\boldsymbol{E}}$. Since for $\widetilde{\boldsymbol{E}} \in X_N$ and $\boldsymbol{H} \in H(\mathbf{curl}\,;\Omega)$, there holds:

$$\int_\Omega \boldsymbol{H} \cdot \mathbf{curl}\,\widetilde{\boldsymbol{E}}\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{H} \cdot \widetilde{\boldsymbol{E}}\,dx,$$

we obtain

$$\boldsymbol{E} \in X_N, \quad \forall \widetilde{\boldsymbol{E}} \in X_N, \quad \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\widetilde{\boldsymbol{E}} - \omega^2\,\varepsilon \boldsymbol{E} \cdot \widetilde{\boldsymbol{E}} = i\omega \int_\Omega \boldsymbol{J} \cdot \widetilde{\boldsymbol{E}}. \tag{1.2}$$

Taking account of the equation (1.1c) on the divergence of $\boldsymbol{E}$, we introduce a parameter $s > 0$ and the new right hand sides

$$\boldsymbol{f}[\boldsymbol{J}, s](\boldsymbol{v}) = i\omega \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} + \frac{s}{i\omega}\int_\Omega \operatorname{div}\boldsymbol{J}\ \operatorname{div}\bar{\varepsilon}\boldsymbol{v} \qquad \text{and} \qquad g[\boldsymbol{J}] = \frac{1}{i\omega}\operatorname{div}\boldsymbol{J}. \tag{1.3}$$

Then we define the following variational problem

$$\boldsymbol{u} \in X_N,\ \forall \boldsymbol{v} \in X_N,\ \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + s\operatorname{div}\varepsilon\boldsymbol{u}\ \operatorname{div}\bar{\varepsilon}\boldsymbol{v} - \omega^2\,\varepsilon\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{f}(\boldsymbol{v}), \tag{1.4}$$

and its saddle-point version, which involves a Lagrange multiplier p (a pseudo-pressure):

$$(\boldsymbol{u}, p) \in X_N \times L^2(\Omega), \quad \forall(\boldsymbol{v}, q) \in X_N \times L^2(\Omega),$$

$$\left\{ \begin{aligned} \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + s\operatorname{div}\varepsilon\boldsymbol{u}\ \operatorname{div}\bar{\varepsilon}\boldsymbol{v} + p\ \operatorname{div}\bar{\varepsilon}\boldsymbol{v} - \omega^2\,\varepsilon\boldsymbol{u}\cdot\boldsymbol{v} &= \boldsymbol{f}(\boldsymbol{v}), \\ \int_\Omega \operatorname{div}\varepsilon\boldsymbol{u}\,q &= \int_\Omega g\,q. \end{aligned} \right. \tag{1.5}$$

The following statement yields the conditions of equivalence between problems (1.1), (1.4) and (1.5) if $\boldsymbol{f}$ and g are defined in (1.3). The essential argument relies on the properties of the operator

$$\begin{array}{rccc} \Delta_\varepsilon^{\mathrm{Dir}}: & \mathring{H}^1(\Omega) & \longrightarrow & H^{-1}(\Omega) \\ & \Phi & \longmapsto & \operatorname{div}\varepsilon\,\mathbf{grad}\,\Phi. \end{array} \tag{1.6}$$

The assumption about ε implies that the sesquilinear form associated with $-\Delta_\varepsilon^{\mathrm{Dir}}$ is coercive on $\mathring{H}^1(\Omega)$:

$$\operatorname{Re}\int_\Omega \varepsilon\,\mathbf{grad}\,\Phi \cdot \overline{\mathbf{grad}\,\Phi} \geq \rho_0 |\Phi|^2_{H^1(\Omega)}.$$

Thus the operator $-\Delta_\varepsilon^{\mathrm{Dir}}$ is invertible from its Dirichlet domain

$$D(\Delta_\varepsilon^{\mathrm{Dir}}) = \{\Phi \in \mathring{H}^1(\Omega) \mid \quad \Delta_\varepsilon^{\mathrm{Dir}}\Phi \in L^2(\Omega)\}$$

onto $L^2(\Omega)$ and has a discrete spectrum.

Theorem 1.1 *We assume $\omega \neq 0$. Let $\boldsymbol{J} \in H(\mathrm{div}\,;\Omega)$; for a fixed $s > 0$, $\boldsymbol{f} = \boldsymbol{f}[\boldsymbol{J}, s]$ and $g = g[\boldsymbol{J}]$ as defined in* (1.3).

(i) *If $(\boldsymbol{E}, \boldsymbol{H})$ solves* (1.1)*, then $\boldsymbol{u} = \boldsymbol{E}$ solves* (1.4) *and $(\boldsymbol{u}, p) = (\boldsymbol{E}, 0)$ solves* (1.5).

(ii) *If $\boldsymbol{u}$ solves* (1.4) *and ω^2/s is not an eigenvalue of the Dirichlet operator $-\Delta_\varepsilon^{\mathrm{Dir}}$ on Ω, then $(\boldsymbol{E}, \boldsymbol{H}) = (\boldsymbol{u}, (i\omega\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{u})$ solves* (1.1).

(iii) *If $(\boldsymbol{u}, p)$ solves* (1.5)*, then $p = 0$ and $(\boldsymbol{E}, \boldsymbol{H}) = (\boldsymbol{u}, (i\omega\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{u})$ solves* (1.1).

PROOF. *(i)* was proved while stating problems (1.4) and (1.5).

(ii) In (1.4) let us take as test functions all fields $\boldsymbol{v} = \mathbf{grad}\,\overline{\Phi}$ with $\Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}})$, which ensures that $\mathbf{grad}\,\Phi \in X_N$. Let us denote by $\langle a, b\rangle_\Omega := \int_\Omega a\,\bar{b}$ the hermitian scalar product on $L^2(\Omega)$. Using the expression (1.3) of the right hand side, and the identities, valid for $\Phi \in \mathring{H}^1(\Omega)$

$$\big\langle \varepsilon\boldsymbol{u}\,, \mathbf{grad}\,\Phi \big\rangle_\Omega = -\big\langle\, \mathrm{div}\,\varepsilon\boldsymbol{u}\,, \Phi \big\rangle_\Omega \quad \text{and} \quad \big\langle \boldsymbol{J}\,, \mathbf{grad}\,\Phi \big\rangle_\Omega = -\big\langle\, \mathrm{div}\,\boldsymbol{J}\,, \Phi \big\rangle_\Omega\,,$$

we easily arrive at

$$\big\langle\, \mathrm{div}\,\varepsilon\boldsymbol{u} - g\,,\; s\Delta_\varepsilon^{\mathrm{Dir}}\Phi + \omega^2\Phi \big\rangle_\Omega = 0$$

for all $\Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}})$. Thus if ω^2/s is not an eigenvalue of $-\Delta_\varepsilon^{\mathrm{Dir}}$, we find that $\mathrm{div}\,\varepsilon\boldsymbol{u} = g$. We deduce that $\boldsymbol{u}$ solves problem (1.2). Thus:

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2\,\varepsilon\boldsymbol{E} = i\omega\boldsymbol{J}.$$

Setting $\boldsymbol{H} = (i\omega\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{u}$, we arrive at (1.1).

(iii) We obtain similarly that

$$\forall\Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle p\,, \Delta_\varepsilon^{\mathrm{Dir}}\Phi \big\rangle_\Omega = 0,$$

whence $p = 0$, since $\Delta_\varepsilon^{\mathrm{Dir}}$ is invertible. Thus $\boldsymbol{u}$ solves (1.2), and $(\boldsymbol{u}, (i\omega\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{u})$ solves (1.1). ■

Both formulations (1.4) and (1.5) are strongly elliptic in the following sense. The norm $\|\cdot\|_{X_N}$ of X_N is given by

$$\|\boldsymbol{u}\|^2_{X_N} = \int_\Omega |\,\mathbf{curl}\,\boldsymbol{u}|^2 + |\,\mathrm{div}\,\varepsilon\boldsymbol{u}|^2 + |\boldsymbol{u}|^2.$$

The principal part of the sesquilinear form associated with problem (1.4)

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u} \cdot \overline{\mathbf{curl}\,\boldsymbol{v}} + s\,\mathrm{div}\,\varepsilon\boldsymbol{u}\;\overline{\mathrm{div}\,\varepsilon\boldsymbol{v}}$$

is coercive on X_N. Moreover, concerning the saddle-point formulation (1.5), we introduce

$$b(p, \boldsymbol{v}) = \int_\Omega p\;\overline{\mathrm{div}\,\varepsilon\boldsymbol{v}}$$

and, as an easy consequence of the invertibility of $\Delta_\varepsilon^{\rm Dir}$, we have the Babuška–Brezzi inf-sup condition, for a constant $\beta > 0$:

$$\forall p \in L^2(\Omega), \qquad \sup_{\boldsymbol{v}\in X_N} \frac{b(p,\boldsymbol{v})}{\|\boldsymbol{v}\|_{X_N}} \geq \beta \, \|p\|_{L^2(\Omega)} .$$

Thus, being in presence of strongly elliptic systems, we can be sure that, with some adaptation due to the non-standard variational space, the classical Mellin approach will be applicable in the situation of smooth coefficients and domains with edges and corners.

1.c Variational formulation for the magnetic field

We describe the situation for the magnetic field $\boldsymbol{H}$ with less details, since there are numerous symmetries and Theorem 1.1 yields already information for $\boldsymbol{H}$.

We have already seen that a suitable variational space for $\boldsymbol{H}$ is X_T. The first variational formulation is obtained by integrating the second equation of (1.1a) versus ${}^{\rm T}\varepsilon^{-1}\widetilde{\boldsymbol{H}}$, and the first one versus $i\omega\widetilde{\boldsymbol{H}}$, for any $\widetilde{\boldsymbol{H}} \in X_T$:

$$\int_\Omega \varepsilon^{-1}\, \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}} - \omega^2\, \mu \boldsymbol{H} \cdot \widetilde{\boldsymbol{H}} = \int_\Omega \varepsilon^{-1} \boldsymbol{J} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}} =: \boldsymbol{h}(\widetilde{\boldsymbol{H}}). \tag{1.7}$$

Taking account of the equation (1.1c) $\operatorname{div} \mu\boldsymbol{H} = 0$, we obtain the following variational problem

$$\boldsymbol{u}\in X_T,\ \forall \boldsymbol{v}\in X_T,\ \int_\Omega \varepsilon^{-1}\, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + s \operatorname{div} \mu \boldsymbol{u}\, \operatorname{div} \bar\mu \boldsymbol{v} - \omega^2\, \mu \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{h}(\boldsymbol{v}), \tag{1.8}$$

and its saddle-point version:

$$(\boldsymbol{u},p) \in X_T \times L^2(\Omega)/\mathbb{C}, \quad \forall (\boldsymbol{v},q) \in X_T \times L^2(\Omega)/\mathbb{C},$$
$$\left\{ \begin{array}{rcl} \displaystyle\int_\Omega \varepsilon^{-1}\, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + s \operatorname{div} \mu \boldsymbol{u}\, \operatorname{div} \bar\mu \boldsymbol{v} + p \, \operatorname{div} \bar\mu \boldsymbol{v} - \omega^2\, \mu \boldsymbol{u} \cdot \boldsymbol{v} & = & \boldsymbol{h}(\boldsymbol{v}), \\ \displaystyle\int_\Omega \operatorname{div} \mu \boldsymbol{u}\, q & = & 0. \end{array} \right. \tag{1.9}$$

The Laplace-like operator which plays a similar role as $\Delta_\varepsilon^{\rm Dir}$ is the Neumann operator $\Delta_\mu^{\rm Neu}$ defined from its domain

$$D(\Delta_\mu^{\rm Neu}) = \{\Phi \in H^1(\Omega) \mid \quad \operatorname{div} \mu\, \mathbf{grad}\, \Phi \in L^2(\Omega) \quad \text{and} \quad \partial_n \Phi = 0 \text{ on } \partial\Omega\}$$

by $\Delta_\mu^{\rm Neu}\Phi = \operatorname{div} \mu\, \mathbf{grad}\, \Phi$. The operator $-\Delta_\mu^{\rm Neu}$ is invertible from $D(\Delta_\mu^{\rm Neu})/\mathbb{C}$ onto $L^2_0(\Omega)$ (the subspace orthogonal to constants) and has a discrete spectrum.

Theorem 1.2 *We assume* $\omega \neq 0$. *Let* $\boldsymbol{J} \in H(\mathrm{div}\,;\Omega)$ *;* $\boldsymbol{h}$ *is defined from* $\boldsymbol{J}$ *in* (1.7). *For a fixed* $s > 0$ *:*

(i) *If* $(\boldsymbol{E}, \boldsymbol{H})$ *solves* (1.1), *then* $\boldsymbol{u} = \boldsymbol{H}$ *solves* (1.8) *and* $(\boldsymbol{u}, p) = (\boldsymbol{H}, c)$ *solves* (1.9) *for any* $c \in \mathbb{C}$.

(ii) *If* $\boldsymbol{u}$ *solves* (1.8) *and* ω^2/s *is not an eigenvalue of* $-\Delta_\mu^{\mathrm{Neu}}$ *on* Ω, *then* $(\boldsymbol{E}, \boldsymbol{H}) = (\frac{i}{\omega}\,\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{J}), \boldsymbol{u})$ *solves* (1.1).

(iii) *If* $(\boldsymbol{u}, p)$ *solves* (1.9), *then* p *is a constant and* $(\frac{i}{\omega}\,\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{J}), \boldsymbol{u})$ *solves* (1.1).

1.d Symmetric roles of divergences and pressures. Regularity

Under a weak assumption of regularity on the right hand sides, we obtain the H^1 regularity for the divergence $\mathrm{div}\,\varepsilon\boldsymbol{u}$ in (1.4) and the pressure p in (1.5). This comes from the fact that $\mathrm{div}\,\varepsilon\boldsymbol{u}$ and p are solutions of independent boundary value problems.

Let $\varepsilon^* = {}^{\mathrm{T}}\bar{\varepsilon}$ be the hermitian adjoint of ε.

Theorem 1.3 *Let* $s > 0$, $\boldsymbol{f} \in L^2(\Omega)^3$ *and* $g \in L^2(\Omega)$. *We assume that* ω^2/s *is not an eigenvalue of the Dirichlet operator* $-\Delta_\varepsilon^{\mathrm{Dir}}$ *on* Ω.

(i) *If* $\boldsymbol{u}$ *solves* (1.4), *then* $\mathrm{div}\,\varepsilon\boldsymbol{u}$ *is the solution* q *of the Dirichlet problem*

$$\left\{\begin{array}{l} q \in \overset{\circ}{H}{}^1(\Omega), \\ s\Delta_{\varepsilon^*}^{\mathrm{Dir}} q + \omega^2 q = -\,\mathrm{div}\,\boldsymbol{f}. \end{array}\right. \tag{1.10}$$

(ii) *If* $(\boldsymbol{u}, p)$ *solves* (1.5), *then* $p + sg$ *is the solution* q *of the Dirichlet problem*

$$\left\{\begin{array}{l} q \in \overset{\circ}{H}{}^1(\Omega), \\ \Delta_{\varepsilon^*}^{\mathrm{Dir}} q = -\,\mathrm{div}\,\boldsymbol{f} - \omega^2 g. \end{array}\right. \tag{1.11}$$

Thus, if moreover g belongs to $H^1(\Omega)$, the pseudo-pressure p belongs to $H^1(\Omega)$ too. As a straightforward consequence of Theorem 1.3, we rediscover the compatibility condition between $\boldsymbol{f}$ and g (compare (1.3)), which insures that $\mathrm{div}\,\varepsilon\boldsymbol{u} = g$ and $p = 0$ respectively:

Corollary 1.4 *Under the assumptions of Theorem* 1.3, *the three following conditions are equivalent:*

(i) g *is solution of the Dirichlet problem* (1.10)*;*

(ii) *If* $\boldsymbol{u}$ *solves* (1.4), *then* $\mathrm{div}\,\varepsilon\boldsymbol{u} = g$ *;*

(iii) *If* $(\boldsymbol{u}, p)$ *solves* (1.5), *then* $p = 0$.

In particular, if we search for divergence-ε free solutions of (1.4), the necessary and sufficient condition on $\boldsymbol{f}$ is that $\boldsymbol{f}$ is divergence free.

Proof of Theorem 1.3. *(i)* If $\boldsymbol{u}$ solves (1.4), then taking as test functions $\boldsymbol{v} = \mathbf{grad}\,\overline{\Phi}$ with $\Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}})$ we obtain

$$\forall \Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle \operatorname{div} \varepsilon \boldsymbol{u}\,,\, s\Delta_\varepsilon^{\mathrm{Dir}}\Phi + \omega^2 \Phi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\Phi \big\rangle_\Omega . \tag{1.12}$$

But the solution of (1.10) satisfies

$$\forall \Psi \in \overset{\circ}{H}{}^1(\Omega), \qquad -\big\langle s\varepsilon^* \,\mathbf{grad}\, q\,,\, \mathbf{grad}\,\Psi \big\rangle_\Omega + \big\langle \omega^2 q\,,\, \Psi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\Psi \big\rangle_\Omega ,$$

whence

$$\forall \Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle q\,,\, s\Delta_\varepsilon^{\mathrm{Dir}}\Phi + \omega^2 \Phi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\Phi \big\rangle_\Omega .$$

Thus $\operatorname{div} \varepsilon \boldsymbol{u} - q$ is orthogonal to the range of $s\Delta_\varepsilon^{\mathrm{Dir}} + \omega^2$, which is the whole $L^2(\Omega)$.

(ii) If $(\boldsymbol{u}, p)$ solves (1.5), we obtain similarly that

$$\forall \Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle sg + p\,,\, \Delta_\varepsilon^{\mathrm{Dir}}\Phi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\Phi \big\rangle_\Omega - \big\langle \omega^2 g\,,\, \Phi \big\rangle_\Omega ,$$

and that the solution of (1.11) satisfies

$$\forall \Phi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle q\,,\, \Delta_\varepsilon^{\mathrm{Dir}}\Phi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\Phi \big\rangle_\Omega - \big\langle \omega^2 g\,,\, \Phi \big\rangle_\Omega .$$

Whence $sg + p - q$ is orthogonal to the range of $\Delta_\varepsilon^{\mathrm{Dir}}$, which is the whole $L^2(\Omega)$. ∎

We have similar statements concerning the "magnetic" problems (1.8) and (1.9): if $\boldsymbol{u}$ solves (1.8), then $\operatorname{div} \mu \boldsymbol{u}$ is the solution in $H^1(\Omega)$ (with zero mean value if $\omega = 0$) of a Neumann problem. If $(\boldsymbol{u}, p)$ solves (1.9), then $p + sg$ is similarly solution of a Neumann problem.

Remark 1.5 Here is another equivalent variational formulation of Maxwell equation. It satisfies an inf-sup condition and is used in finite-element approximations, see Nedelec [20], Girault - Raviart [12] and also [1].

Let $\overset{\circ}{H}(\mathbf{curl};\Omega)$ be the space of the fields $\boldsymbol{u} \in H(\mathbf{curl};\Omega)$ such that $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega$. With this definition, we can introduce the alternative saddle-point version, which is an ersatz for (1.5) when $s = 0$:

$$(\boldsymbol{u}, p) \in \overset{\circ}{H}(\mathbf{curl};\Omega) \times \overset{\circ}{H}{}^1(\Omega), \quad \forall (\boldsymbol{v}, q) \in \overset{\circ}{H}(\mathbf{curl};\Omega) \times \overset{\circ}{H}{}^1(\Omega),$$
$$\left\{ \begin{array}{rcl} \displaystyle\int_\Omega \mu^{-1}\, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} - \mathbf{grad}\, p \cdot \bar\varepsilon \boldsymbol{v} - \omega^2\, \varepsilon \boldsymbol{u} \cdot \boldsymbol{v} &=& \boldsymbol{f}_0(\boldsymbol{v}), \\ \displaystyle -\int_\Omega \varepsilon \boldsymbol{u}\ \mathbf{grad}\, q &=& \displaystyle\int_\Omega g\, q. \end{array} \right. \tag{1.13}$$

For any $s > 0$ and $\boldsymbol{v} \in X_N$, we define, *cf* (1.3)

$$\boldsymbol{f}_s(\boldsymbol{v}) = \boldsymbol{f}_0(\boldsymbol{v}) + s \int_\Omega g \ \operatorname{div} \bar\varepsilon \boldsymbol{v}.$$

Then, if $\boldsymbol{f}_0 \in L^2(\Omega)^3$ and $g \in \mathring{H}^1(\Omega)$, we can prove, as a consequence of Theorem 1.3 and of the density of $\mathscr{D}(\Omega)^3$ in $\mathring{H}(\mathbf{curl};\Omega)$ that $(\boldsymbol{u},p)$ solves (1.13) if and only if $(\boldsymbol{u},p)$ solves (1.5) with $\boldsymbol{f} = \boldsymbol{f}_s$. ■

2 Second order elliptic version of Maxwell's equations

As a second step towards the analysis of singularities, we are going to formulate problems (1.4) and (1.5) (or (1.8) and (1.9)) as elliptic boundary value problems. Here we leave the most general framework, and assume that ε and μ are piecewise smooth, the regions Ω_j, $j = 1,\dots,J$, where they are smooth forming a polyhedral partition of the domain Ω which is supposed itself to be pseudo-polyhedral in the sense of the definition in subsection 0.c. We denote by F_{jk} the (open) faces of Ω_j, by $\mathscr{F}$ the set of the interior faces (the F_{jk} inside Ω) and by $\mathscr{F}_0$ the set of the faces contained in $\partial\Omega$.

As our aim is to concentrate on the singularities of the solutions, we can now take advantage of elliptic theory and drop the non-principal parts in the elliptic variational problems (1.4) and (1.5), which corresponds to setting $\omega = 0$. We fix also $s = 1$. In order to give sense to *natural* boundary conditions, we also take more regular right hand sides: $\boldsymbol{f}$ or $\boldsymbol{h} \in L^2(\Omega)^3$ and $g \in H^1(\Omega)$.

2.a Transmission problems for the electric field

Let us begin with problem (1.4), which is now

$$\boldsymbol{u} \in X_N, \quad \forall \boldsymbol{v} \in X_N, \quad \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \operatorname{div}\varepsilon\boldsymbol{u}\,\operatorname{div}\bar{\varepsilon}\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}, \tag{2.1}$$

with $\boldsymbol{f} \in L^2(\Omega)^3$. This problem has a unique solution.

The conversion of (2.1) to a boundary value problem is obtained by a judicious choice of test functions. We first note that if ε is discontinuous, the space X_N does not contain $\mathscr{C}^\infty(\overline{\Omega})^3$, but the piecewise smooth fields $\boldsymbol{v}$, whose restrictions to Ω_j are denoted $\boldsymbol{v}_j$, satisfying:

$$\begin{cases} \boldsymbol{v}_j \in \mathscr{C}^\infty(\overline{\Omega}_j)^3, \quad j = 1,\dots,J \\ [\boldsymbol{v}\times\boldsymbol{n}]_F = 0, \ \forall F \in \mathscr{F} \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n}\big|_F = 0, \ \forall F \in \mathscr{F}_0, \\ [(\varepsilon\boldsymbol{v})\cdot\boldsymbol{n}]_F = 0, \ \forall F \in \mathscr{F}, \end{cases} \tag{2.2}$$

where the jump $[\boldsymbol{v}\times\boldsymbol{n}]_F$ is equal to $(\boldsymbol{v}_j \times \boldsymbol{n}_j - \boldsymbol{v}_{j'} \times \boldsymbol{n}_j)\big|_F$ if F belongs to $\partial\Omega_j$ and to $\partial\Omega_{j'}$ and with $\boldsymbol{n}_j$ the exterior unit normal to $\partial\Omega_j$; $[\boldsymbol{w}\cdot\boldsymbol{n}]_F$ is defined similarly.

Taking as test functions $\boldsymbol{v}$ such that $\boldsymbol{v}_j$ belongs to $\mathscr{C}_0^\infty(\Omega_j)^3$, we obtain the interior equations (with $\varepsilon_j^* := {}^{\mathrm{T}}\bar{\varepsilon}_j$)

$$\mathbf{curl}\,\mu_j^{-1}\,\mathbf{curl}\,\boldsymbol{u}_j - \varepsilon_j^*\,\mathbf{grad}\,\operatorname{div}\varepsilon_j\boldsymbol{u}_j = f_j \quad \text{in} \quad \Omega_j, \quad j = 1,\dots,J. \tag{2.3a}$$

The stable boundary conditions are those contained in (2.2):

$$\begin{cases} [\boldsymbol{u}\times\boldsymbol{n}]_F = 0, \ \ \forall F\in\mathscr{F} \quad \text{and} \quad \boldsymbol{u}\times\boldsymbol{n}\big|_F = 0, \ \ \forall F\in\mathscr{F}_0, \\ [(\varepsilon\boldsymbol{u})\cdot\boldsymbol{n}]_F = 0, \ \ \forall F\in\mathscr{F}. \end{cases} \tag{2.3b}$$

The natural boundary conditions are obtained by the following choices of test functions: For each $F = F_{jk}\in\mathscr{F}_0$ we take any function $\varphi\in\mathscr{C}_0^\infty(F)$ and a field $\boldsymbol{v}$ satisfying (2.2) with $\varepsilon_j\boldsymbol{v}_j\cdot\boldsymbol{n}_j\big|_F = \varphi$ and $\boldsymbol{v}=0$ on all the other faces. For each $F = F_{jk} = F_{j'k'}\in\mathscr{F}$ we take any field $\boldsymbol{\varphi}\in\mathscr{C}_0^\infty(F)^3$ and a field $\boldsymbol{v}$ satisfying (2.2) with $(\varepsilon_j\boldsymbol{v}_j)\cdot\boldsymbol{n}_j\big|_F = \boldsymbol{\varphi}\cdot\boldsymbol{n}_j$, $\boldsymbol{v}_j\times\boldsymbol{n}_j\big|_F = \boldsymbol{\varphi}\times\boldsymbol{n}_j$ and $\boldsymbol{v}=0$ on all the other faces. Thus we obtain

$$\begin{cases} [\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}]_F = 0, \ \ \forall F\in\mathscr{F} \\ [\operatorname{div}\varepsilon\boldsymbol{u}]_F = 0, \ \ \forall F\in\mathscr{F} \quad \text{and} \quad \operatorname{div}\varepsilon\boldsymbol{u}\big|_F = 0, \ \ \forall F\in\mathscr{F}_0. \end{cases} \tag{2.3c}$$

The three above equations have to be completed by the basic regularity conditions (*cf* Theorem 1.3)

$$\boldsymbol{u}\in L^2(\Omega)^3, \quad \mathbf{curl}\,\boldsymbol{u}\in L^2(\Omega)^3 \quad \text{and} \quad \operatorname{div}\varepsilon\boldsymbol{u}\in H^1(\Omega). \tag{2.3d}$$

Concerning the saddle-point problem (1.5), the situation is completely similar. The simplified form of the problem is

$$(\boldsymbol{u},p)\in X_N\times L^2(\Omega), \quad \forall(\boldsymbol{v},q)\in X_N\times L^2(\Omega),$$
$$\begin{cases} \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \operatorname{div}\varepsilon\boldsymbol{u}\ \operatorname{div}\bar\varepsilon\boldsymbol{v} + p\ \operatorname{div}\bar\varepsilon\boldsymbol{v} &= \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}, \\ \displaystyle\int_\Omega \operatorname{div}\varepsilon\boldsymbol{u}\,q &= \displaystyle\int_\Omega g\,q. \end{cases} \tag{2.4}$$

2.b Transmission problems for the magnetic field

As for problem (1.8), we write its principal part as follows

$$\boldsymbol{u}\in X_T, \quad \forall\boldsymbol{v}\in X_T, \quad \int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \operatorname{div}\mu\boldsymbol{u}\ \operatorname{div}\bar\mu\boldsymbol{v} = \int_\Omega \boldsymbol{h}\cdot\boldsymbol{v}, \tag{2.5}$$

with $\boldsymbol{h}\in L^2(\Omega)^3$. This problem has a unique solution.

The conversion of (2.5) into a transmission problem is performed in the same way as before.

The interior equations are:

$$\mathbf{curl}\,\varepsilon_j^{-1}\,\mathbf{curl}\,\boldsymbol{u}_j - \mu_j^*\,\mathbf{grad}\operatorname{div}\mu_j\boldsymbol{u}_j = f_j \quad \text{in} \quad \Omega_j, \quad j=1,\dots,J. \tag{2.6a}$$

The stable boundary conditions are:

$$\begin{cases} [\boldsymbol{u}\times\boldsymbol{n}]_F = 0, \ \ \forall F\in\mathscr{F}, \\ [(\mu\boldsymbol{u})\cdot\boldsymbol{n}]_F = 0, \ \ \forall F\in\mathscr{F} \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{n}\big|_F = 0, \ \ \forall F\in\mathscr{F}_0. \end{cases} \tag{2.6b}$$

The natural boundary conditions are:

$$\begin{cases} [\varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}]_F = 0, \ \ \forall F\in\mathscr{F} \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\big|_F = 0, \ \ \forall F\in\mathscr{F}_0 \\ [\operatorname{div}\mu\boldsymbol{u}]_F = 0, \ \ \forall F\in\mathscr{F}. \end{cases} \tag{2.6c}$$

The three above equations have to be completed by the basic regularity conditions

$$\boldsymbol{u}\in L^2(\Omega)^3, \quad \mathbf{curl}\,\boldsymbol{u}\in L^2(\Omega)^3 \quad \text{and} \quad \operatorname{div}\mu\boldsymbol{u}\in H^1(\Omega). \tag{2.6d}$$

Concerning the saddle-point problem (1.9), the situation is completely similar.

2.c Variational problems in H^1

In this paragraph we assume that the set $\mathscr{F}$ of interfaces is empty, that is, ε and μ are smooth throughout $\overline{\Omega}$. The variational problem (2.1) is then just a boundary value problem: Equation (2.3a) with boundary conditions

$$\boldsymbol{u}\times\boldsymbol{n} = 0 \quad \text{and} \quad \operatorname{div}\varepsilon\boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega.$$

This same boundary value problem can be formulated in a new variational space with more regularity:

$$H_N = X_N\cap H^1(\Omega)^3 = \{\boldsymbol{u}\in H^1(\Omega)^3 \mid \quad \boldsymbol{u}\times\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\}. \tag{2.7}$$

In [8] is proved that H_N is a closed subspace of X_N for its norm $\|\cdot\|_{X_N}$. Thus the variant of problem (2.1)

$$\boldsymbol{u}\in H_N, \quad \forall\boldsymbol{v}\in H_N, \quad \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \operatorname{div}\varepsilon\boldsymbol{u}\,\operatorname{div}\bar\varepsilon\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}, \tag{2.8}$$

has a unique solution too. If $\boldsymbol{u}$ is solution of this problem, $\boldsymbol{u}$ also satisfies (2.3a)-(2.3c), but not (2.3d): more precisely there is no reason why $\operatorname{div}\varepsilon\boldsymbol{u}\in H^1(\Omega)$ should hold: In the analogue of the proof of Theorem 1.3 *(i)* we find that now we have to take the test function Φ in $D(\Delta_\varepsilon^{\mathrm{Dir}})\cap H^2(\Omega)$, hence $\operatorname{div}\varepsilon\boldsymbol{u}$ is the solution q of the "very weak" form of the Dirichlet problem:

$$\begin{cases} q\in L^2(\Omega), \\ \forall\Phi\in \mathring{H}^1(\Omega)\cap H^2(\Omega), \qquad \big\langle q\,,\,\Delta_\varepsilon^{\mathrm{Dir}}\Phi\big\rangle_\Omega = -\big\langle \operatorname{div}\boldsymbol{f}\,,\,\Phi\big\rangle_\Omega\,. \end{cases} \tag{2.9}$$

Contrary to (1.12), (2.9) does not imply $q\in H^1(\Omega)$, nor does it have a unique solution, if Ω is not convex. In particular, if $\operatorname{div}\boldsymbol{f} = 0$, then the solution $\boldsymbol{u}\in X_N$ of (2.1) satisfies $\operatorname{div} \varepsilon \boldsymbol{u} = 0$. On the other hand, if if Ω has non-convex edges, the solution space of (2.9) is infinite-dimensional, $\operatorname{div} \varepsilon \boldsymbol{u}$ is a dual singular function for the Dirichlet problem, and the equivalence with the original Maxwell equations does not hold.

Concerning the saddle-point problem (1.5), we may consider the corresponding problem set in $H_N \times L^2(\Omega)$. Here again, after integration by parts versus suitable test functions, the difference between the formulations in X_N and H_N appear in the regularity properties of the solutions (we assume $\boldsymbol{f} \in L^2(\Omega)^3$ and $g \in H^1(\Omega)$): then $(\boldsymbol{u}, p) \in X_N \times H^1(\Omega)$ for problem (2.4) and $(\boldsymbol{u}, p) \in H_N \times L^2(\Omega)$ for its H^1-type counterpart.

The corresponding variational space for the magnetic problem is:

$$H_T = X_T \cap H^1(\Omega)^3 = \{\boldsymbol{u} \in H^1(\Omega)^3 \mid \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\}. \tag{2.10}$$

We have the corresponding variant of problem (2.5)

$$\boldsymbol{u} \in H_T, \quad \forall \boldsymbol{v} \in H_T, \quad \int_\Omega \varepsilon^{-1} \operatorname{\mathbf{curl}} \boldsymbol{u} \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} + \operatorname{div} \mu \boldsymbol{u} \, \operatorname{div} \bar{\mu} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{2.11}$$

which uniquely solvable too.

3 Regularity and singularities on polygons

Although our final aim would be to analyze the singularities of *transmission problems* as described in the previous section, here we consider only the case of *boundary value problems*. Thus we shall leave to a forthcoming paper the investigation of the regularity and the singular parts of the solutions of problems (1.4) and (1.8) (or equivalently of their saddle-point versions (1.5) and (1.9)), when the domain Ω belongs to the class of polyhedral domains and is formed by the juxtaposition of several materials occupying themselves polyhedral subdomains Ω_j, which corresponds to ε and μ piecewise regular (smooth on each Ω_j).

In this paper, we concentrate now on the following particular situation, which is however a good representative of the general case:

- Ω is pseudo-polygonal or pseudo-polyhedral,
- ε and μ are equal to the identity matrix,
- $s = 1$ and $\omega = 0$ as explained in the last section.

Classically, the properties of the solution are determined in each singular point x of the unfolded boundary $\partial\Omega$ by singular exponents $\lambda \in \mathbb{C}$, corresponding themselves to singular functions behaving roughly as d_x^λ (with d_x the distance to x). The aim of the next sections is the description of the singular exponents and functions.

For the sake of simplicity we begin with Maxwell equations in $\mathbb{R}^2$ which are obtained from the three-dimensional ones by the elimination of one coordinate (say the third one) and of the corresponding component in the fields. The curl has now two (dual) forms, one scalar when applied to 2D fields $\operatorname{curl} \boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1$, and one vectorial when applied to scalar functions $\mathbf{curl}\, w = {}^{\mathrm{T}}(\partial_2 w, -\partial_1 w)$. The definitions of the divergence and gradient are obvious: $\operatorname{div} \boldsymbol{v} = \partial_1 v_1 + \partial_2 v_2$, and $\mathbf{grad}\, w = {}^{\mathrm{T}}(\partial_1 w, \partial_2 w)$. The domain Ω is supposed to be pseudo-polygonal and the space X_N is then the space of $L^2(\Omega)^2$ fields $\boldsymbol{v}$ such that $\operatorname{curl} \boldsymbol{v}$ and $\operatorname{div} \boldsymbol{v}$ are $L^2(\Omega)$, and with tangential boundary conditions.

Thus the "electric" problem under consideration is:

$$\boldsymbol{u} \in X_N, \quad \forall \boldsymbol{v} \in X_N, \quad \int_\Omega \operatorname{curl} \boldsymbol{u} \ \operatorname{curl} \boldsymbol{v} + \operatorname{div} \boldsymbol{u} \ \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{3.1}$$

and the corresponding boundary value problem is

$$\begin{cases} \mathbf{curl} \operatorname{curl} \boldsymbol{u} - \mathbf{grad} \operatorname{div} \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{on } \partial\Omega, \\ \boldsymbol{u} \in L^2(\Omega)^2, \quad \operatorname{curl} \boldsymbol{u} \in L^2(\Omega) & \text{and} \quad \operatorname{div} \boldsymbol{u} \in H^1(\Omega). \end{cases} \tag{3.2}$$

We are going to investigate the regularity of the solutions $\boldsymbol{u}$ in the scale of Sobolev spaces when the field $\boldsymbol{f}$ itself belongs to a Sobolev space $H^{s-1}(\Omega)^2$, with $s \geq 1$. Since the boundary value problem (3.2) is an elliptic system, the solution $\boldsymbol{u}$ belongs to $H^{s+1}(\mathscr{V} \cap \Omega)^2$ for any neighborhood $\mathscr{V}$ s.t. $\overline{\mathscr{V}}$ does not meet any corner of Ω. The singular behavior of $\boldsymbol{u}$ is attached to corners.

3.a Corners

Let $\boldsymbol{o}$ be an element of the unfolded boundary of Ω, corresponding to a point $x \in \overline{\Omega} \setminus \Omega$. Thus $\boldsymbol{o}$ belongs to the boundary of one of the $\Omega_{x,i}$ and we say that $\boldsymbol{o}$ is a corner of Ω if the corresponding sector $\Gamma_{x,i}$ is non-trivial (opening $\neq \pi$) and we denote $\Gamma_{x,i}$ by $\Gamma_{\boldsymbol{o}}$ and $\Omega_{x,i}$ by $\mathscr{V}_{\boldsymbol{o}}$. Let $\mathscr{O}$ be the set of the corners $\boldsymbol{o}$ of Ω. At each corner $\boldsymbol{o}$, we associate local polar coordinates such that

$$\Gamma_{\boldsymbol{o}} = \{(r_{\boldsymbol{o}}, \theta_{\boldsymbol{o}}) \mid r_{\boldsymbol{o}} > 0,\ 0 < \theta_{\boldsymbol{o}} < \omega_{\boldsymbol{o}}\} \quad \text{with } \omega_{\boldsymbol{o}} \text{ the opening of } \Gamma_{\boldsymbol{o}}.$$

The regularity and the structure of the solution $\boldsymbol{u}$ only depend on special "eigen-spaces" of pseudo-homogeneous functions $\boldsymbol{Z}(\Gamma_{\boldsymbol{o}})$ defined below.

3.b Pseudo-homogeneous functions

If Γ is a plane sector and $\lambda \in \mathbb{C}$, let $\boldsymbol{S}^\lambda(\Gamma)$ be the space

$$\boldsymbol{S}^\lambda(\Gamma) = \Big\{ \boldsymbol{u} \in X_N^{\mathrm{loc}}(\Gamma^*) \mid \ \boldsymbol{u}(x) = r^\lambda \sum_{q=0}^{Q} \log^q r \ \boldsymbol{v}_q(\theta) \Big\}, \tag{3.3}$$

where $\boldsymbol{u} \in X_N^{\mathrm{loc}}(\Gamma^*)$ means that for any bounded open set $\mathscr{V}$ such that $0 \notin \overline{\mathscr{V}}$, $\boldsymbol{u}$ belongs to $X_N(\Gamma \cap \mathscr{V})$. Thus $\boldsymbol{u} \times \boldsymbol{n}$ is zero on $\partial\Gamma$. The singularities arising at the corner of Γ belong to finite-dimensional subspaces $\boldsymbol{Z}^\lambda(\Gamma)$ of $\boldsymbol{S}^\lambda(\Gamma)$, with λ belonging to a discrete set $\Lambda(\Gamma)$ that we are going to introduce now.

We first need the auxiliary space $\boldsymbol{Y}^\lambda(\Gamma)$ defined as the subspace of $\boldsymbol{S}^\lambda(\Gamma)$:

$$\boldsymbol{Y}^\lambda(\Gamma) = \Big\{\boldsymbol{u} \in \boldsymbol{S}^\lambda(\Gamma) \mid \quad \begin{aligned}&\operatorname{div}\boldsymbol{u} = 0 \ \text{ on } \ \partial\Gamma, \\ &\mathbf{curl}\operatorname{curl}\boldsymbol{u} - \mathbf{grad}\operatorname{div}\boldsymbol{u} \ \text{ is polynomial}\Big\}.\end{aligned} \tag{3.4}$$

If λ is a positive integer, $\boldsymbol{Y}^\lambda(\Gamma)$ contains the space $\boldsymbol{P}^\lambda(\Gamma)$ of homogeneous polynomial (thus non-singular) fields $\boldsymbol{u}$ of degree λ satisfying the boundary conditions $\boldsymbol{u}\times\boldsymbol{n} = 0$ and $\operatorname{div}\boldsymbol{u} = 0$. That is why we introduce a complement $\boldsymbol{Z}^\lambda(\Gamma)$ of $\boldsymbol{P}^\lambda(\Gamma)$ in $\boldsymbol{Y}^\lambda(\Gamma)$

$$\boldsymbol{Y}^\lambda(\Gamma) = \boldsymbol{Z}^\lambda(\Gamma) \oplus \boldsymbol{P}^\lambda(\Gamma). \tag{3.5}$$

Of course, if λ is not a positive integer, $\boldsymbol{Z}^\lambda(\Gamma)$ is simply

$$\Big\{\boldsymbol{u} \in \boldsymbol{S}^\lambda(\Gamma) \mid \ \operatorname{div}\boldsymbol{u} = 0 \text{ on } \partial\Gamma, \ \ \mathbf{curl}\operatorname{curl}\boldsymbol{u} - \mathbf{grad}\operatorname{div}\boldsymbol{u} = 0\Big\}.$$

We denote by $\Lambda(\Gamma)$ the set of $\lambda \in \mathbb{C}$ such that $\boldsymbol{Z}^\lambda(\Gamma)$ is not reduced to $\{0\}$.

3.c Explicit form of singularities

We want to give now the set of exponents and a basis of singular functions for any sector Γ with opening $\omega \neq \pi$. We postpone the proofs to section 5 where the edge singularities are treated. As the two-dimensional corner singularities are a part of them, we can extract from Lemma 5.1 the relevant results.

Lemma 3.1 *Let Γ be a plane sector of opening $\omega \neq \pi$ and let $\Lambda^{\mathrm{Dir}}(\Gamma)$ be the set of Laplace Dirichlet exponents on Γ:*

$$\Lambda^{\mathrm{Dir}}(\Gamma) = \{\tfrac{k\pi}{\omega},\ k \in \mathbb{Z}^*\} \ \ \textit{if } \omega \neq 2\pi \quad \textit{and} \quad \Lambda^{\mathrm{Dir}}(\Gamma) = \{\tfrac{k}{2},\ k \textit{ odd}\} \ \ \textit{if } \omega = 2\pi.$$

Then the corresponding set of (electric) Maxwell exponents is

$$\Lambda(\Gamma) = \{\lambda \in \mathbb{R}\setminus\{1\} \mid \quad \lambda - 1 \ \ \textit{or } \lambda + 1 \textit{ belongs to } \Lambda^{\mathrm{Dir}}(\Gamma)\}.$$

The corresponding spaces of singular functions are generated:

(i) For $\lambda + 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$:
if $\lambda \notin \mathbb{N}$, by $(r^\lambda \sin\lambda\theta,\ r^\lambda\cos\lambda\theta)$,
if $\lambda \in \mathbb{N}$, by $(r^\lambda(\log r\sin\lambda\theta + \theta\cos\theta),\ r^\lambda(\log r\cos\lambda\theta - \theta\sin\theta))$.
These functions are the gradients of the Dirichlet singular functions of the Laplace operator, thus have zero curls and regular divergences.

(ii) For $\lambda - 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$:
if $\lambda \notin \mathbb{N}$, by $(r^\lambda \sin\lambda\theta,\ -r^\lambda\cos\lambda\theta)$,
if $\lambda \in \mathbb{N}$, by $(r^\lambda(\log r\sin\lambda\theta + \theta\cos\theta),\ -r^\lambda(\log r\cos\lambda\theta - \theta\sin\theta))$.
The divergences of these functions are the Dirichlet singular functions of the Laplace operator.

3.d Regularity and singularities in X

From now on, let s denote the regularity exponent. We assume $s \geq 1$. For each $\boldsymbol{o} \in \mathscr{O}$ introduce the following subset of *admissible exponents* $\Lambda^s_{(X)}(\Gamma_{\boldsymbol{o}})$ which correspond to *admissible singular functions* contained in the "energy space", *cf* (2.3d) and which do not belong to $H^{s+1}(\Omega)^2$.

Let $\chi_{\boldsymbol{o}} = \chi(\rho_{\boldsymbol{o}})$ be a smooth cut-off function with its support in $\mathscr{V}_{\boldsymbol{o}}$ and equal to 1 in a neighborhood of $\boldsymbol{o}$. We set

$$\begin{aligned}\Lambda^s_{(X)}(\Gamma_{\boldsymbol{o}}) = \Big\{\lambda \in \Lambda(\Gamma_{\boldsymbol{o}}) \mid \quad & -1 < \operatorname{Re}\lambda \leq s, \\ & \exists \boldsymbol{u} \in \boldsymbol{Z}^\lambda(\Gamma_{\boldsymbol{o}}),\ \boldsymbol{u} \neq 0,\ \ \chi_{\boldsymbol{o}}\boldsymbol{u} \in X_N(\Gamma_{\boldsymbol{o}}),\ \operatorname{div}(\chi_{\boldsymbol{o}}\boldsymbol{u}) \in H^1(\Gamma_{\boldsymbol{o}})\Big\}.\end{aligned}$$

From Lemma 3.1, we can deduce that

$$\begin{aligned}\Lambda^s_{(X)}(\Gamma) \;=\; & \Big\{\mu - 1 \mid \quad \mu \in (0, s+1) \cap \Lambda^{\mathrm{Dir}}(\Gamma)\Big\} \\ & \qquad \bigcup \quad \Big\{\mu + 1 \mid \quad \mu \in (0, s-1) \cap \Lambda^{\mathrm{Dir}}(\Gamma)\Big\}\end{aligned} \tag{3.6}$$

and that for any $\lambda \in \Lambda^s_{(X)}(\Gamma_{\boldsymbol{o}})$, the space $\boldsymbol{Z}^\lambda(\Gamma_{\boldsymbol{o}})$ is one or two-dimensional: let $\boldsymbol{U}^{\lambda,\pm}_{\boldsymbol{o}}$ be the corresponding generators.

Here follows the statement of regularity and singularity for problem (3.1).

Theorem 3.2 *Let* $s \geq 1$ *and let the data* $\boldsymbol{f}$ *belong to* $H^{s-1}(\Omega)^2$.
(i) *If for any* $\boldsymbol{o} \in \mathscr{O}$, $\Lambda^s_{(X)}(\Gamma_{\boldsymbol{o}})$ *is empty, then the solution of problem* (3.1) *belongs to* $H^{s+1}(\Omega)^2$.
(ii) *If for any* $\boldsymbol{o} \in \mathscr{O}$, $\Lambda^s_{(X)}(\Gamma_{\boldsymbol{o}})$ *does not meet the straight line* $\operatorname{Re}\lambda = s$, *then the solution* $\boldsymbol{u}$ *of problem* (3.1) *can be split into*

$$\boldsymbol{u} \;=\; \boldsymbol{u}_{\mathrm{reg}} \quad + \sum_{o\in\mathscr{O},\ \ \lambda\in\Lambda^s_{(X)}(\Gamma_o)} \gamma^{\lambda,\pm}_{\boldsymbol{o}}\ \chi_{\boldsymbol{o}}(r_{\boldsymbol{o}})\ \boldsymbol{U}^{\lambda,\pm}_{\boldsymbol{o}}(r_{\boldsymbol{o}}, \theta_{\boldsymbol{o}}) \tag{3.7}$$

with $\boldsymbol{u}_{\mathrm{reg}} \in H^{s+1}(\Omega)^2$ *and the coefficients* $\gamma^{\lambda,\pm}_{\boldsymbol{o}} \in \mathbb{C}$.

3.e Different choices of regular and singular parts

Theorem 3.2 shows the existence of a splitting into singular functions and a regular part that is as regular as desired. If the singular functions are not constructed according to our explicit formulas in Lemma 3.1, then additional singular terms can be exchanged between the "singular" and "regular" parts: we describe this phenomenon for the simple but important case of the first singularity in $X_N \setminus H_N$ where, according to (2.7), H_N is the space $\{\boldsymbol{v} \in H^1(\Omega)^2,\ \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

Let us consider the expansion (3.7) for $s = 1$: only one singular function $\boldsymbol{U}^{\pi/\omega_o - 1}_{\boldsymbol{o}}$ for each reentrant corner does not belong to H^1. If we put the other terms of the expansion (3.7) into the regular part, we obtain an expansion of the type

$$\boldsymbol{u} \;=\; \boldsymbol{u}_{(H)} \quad + \sum_{o\in\mathscr{O},\ \ o \text{ non-convex}} \gamma_{\boldsymbol{o}}\ \boldsymbol{w}_{\boldsymbol{o}}, \tag{3.8}$$

with $\boldsymbol{w_o} \in X_N \setminus H_N$ and $\boldsymbol{u}_{(H)} \in H_N$.

The question of the regularity of this pseudo regular part $\boldsymbol{u}_{(H)}$ is an important problem if one wants to use the splitting (3.8) for an approximation of $\boldsymbol{u}$ by a singular function method, that is, by trial functions that are composed of the singular functions $\boldsymbol{w_o}$ and regular functions (e.g. piecewise polynomials). The convergence rate of the whole method is then determined by the regularity of $\boldsymbol{u}_{(H)}$.

We compare five constructions for $\boldsymbol{w_o}$ and $\boldsymbol{u}_{(H)}$. In each case, $\boldsymbol{u}_{(H)}$ will have a decomposition itself, and its regularity is determined by its first singular function. For simplicity, we assume that there is just one reentrant corner of opening $\omega > \pi$ situated at the origin 0 and χ_0 denotes a smooth cut-off function equal to 1 in a neighborhood of 0 .

In this case, $\boldsymbol{u}_{(H)} \in H^{\sigma+1}$ with $\sigma < \lambda^*$, where λ^* is the exponent of the first singular function in $\boldsymbol{u}_{(H)}$. Thus λ^* will directly yield the convergence rate of a singular function method for the approximation of $\boldsymbol{u}$.

(i) According to Lemma 3.1, the natural choice for $\boldsymbol{w}$ is $\mathbf{grad}(\chi_0\, r^{\pi/\omega} \sin\frac{\pi\theta}{\omega})$. In this case, the next exponent in $\Lambda^s_{(X)}(\Gamma)$ is $\lambda^* = \frac{2\pi}{\omega} - 1$ if $\omega \neq 2\pi$, and $\lambda^* = \frac{3\pi}{\omega} - 1 = \frac{1}{2}$ if $\omega = 2\pi$. Thus λ^* spans the whole interval $(0, 1)$.

(ii) We can choose a divergence free form of $\boldsymbol{w}$: $\mathbf{curl}(\chi_0\, r^{\pi/\omega} \cos\frac{\pi\theta}{\omega})$. Indeed, this expansion is identical to *(i)*, so λ^* is the same.

(iii) A more abstract construction for $\boldsymbol{w}$ is described in [19] and [5]. Let $\Phi \in \mathring{H}^1(\Omega)$ be the solution of the problem $\Delta\Phi = S_D$, where S_D is the first dual singular function of the Dirichlet problem, i.e. $S_D \neq 0$ satisfies the orthogonality condition

$$\int_\Omega S_D\, \Delta\psi = 0 \quad \forall \psi \in \mathring{H}^1(\Omega) \cap H^2(\Omega).$$

Then the choice of $\boldsymbol{w} = \mathbf{grad}\,\Phi$ provides a splitting of $\boldsymbol{u}$ where $\boldsymbol{w}$ is orthogonal in X_N to the curl-free fields in H_N . Thus if $\mathbf{curl}\,\boldsymbol{f} = 0$, this is an orthogonal decomposition within the spaces of curl-free fields.
It is well known that S_D has a singular part $r^{-\pi/\omega} \sin\frac{\pi\theta}{\omega}$. Therefore, besides the main part $c_0\, r^{\pi/\omega} \sin\frac{\pi\theta}{\omega}$ with non-zero c_0 , Φ contains a singularity of exponent $2 - \frac{\pi}{\omega}$. Thus $\boldsymbol{w}$ contains a term of exponent $1 - \frac{\pi}{\omega}$. For $\omega \neq \frac{3\pi}{2}, 2\pi$, this does not belong to $\Lambda^s_{(X)}$. Thus $\boldsymbol{u}_{(H)}$ must contain this singularity, too, and we have

$$\lambda^* = \min\{\tfrac{2\pi}{\omega} - 1,\ 1 - \tfrac{\pi}{\omega}\} \in (0, \tfrac{1}{3}].$$

This is less regular than the choice *(i)* if $\omega < \frac{3\pi}{2}$.

(iv) A similar construction in [19] and [2] is $\boldsymbol{w} = \mathbf{curl}\,\Phi$ with $\Phi \in H^1(\Omega)$, $\partial_n\Phi = 0$, a solution of $\Delta\Phi = S_N$, S_N the first dual singularity of the Neumann problem. This gives a splitting with $\boldsymbol{w}$ orthogonal in X_N to the divergence-free fields in H_N . Thus if $\operatorname{div} \boldsymbol{f} = 0$, this is an orthogonal decomposition within the spaces of divergence-free

fields. This singular function $\boldsymbol{w}$ is linearly independent of the one in *(iii)*. But modulo $H^1(\Omega)$, these two functions are proportional, and their λ^* are also the same.

(v) Another natural construction is the orthogonal decomposition of $\boldsymbol{u}$ with respect to the inner product in X_N, with the part $\boldsymbol{u}_{(H)}$ in H_N and the residual $\gamma_0\boldsymbol{w}$ in $H_N^\perp$. Thus $\boldsymbol{u}_{(H)}$ is the solution of the variational problem in H_N and $\gamma_0\boldsymbol{w}$ is the difference between these two solutions of the same Maxwell boundary value problem. In this case, $\boldsymbol{u}_{(H)}$ contains a singularity of exponent $1-\frac{\pi}{\omega}$, see (3.9) just below. Thus λ^* is the same as in *(iii)* and *(iv)*. Note that even if $\mathbf{curl}\,\boldsymbol{f}=0$ or $\operatorname{div}\boldsymbol{f}=0$, this decomposition is in general, *not the same* as the one in *(iii)* or *(iv)*, respectively.

3.f Singularities in H^1. Comparison

If instead of (3.1), we consider the variational problem in H_N, the corresponding boundary value problem is the same as (3.2) except the regularity requirements: we have now $\boldsymbol{u}\in H^1(\Omega)^2$ and no special regularity on $\operatorname{div}\boldsymbol{u}$, which is only $L^2(\Omega)$. The corresponding admissible sets of exponents are

$$\Lambda^s_{(H)}(\Gamma_{\boldsymbol{o}})=\Big\{\lambda\in\Lambda(\Gamma_{\boldsymbol{o}})\ \big|\ -1<\operatorname{Re}\lambda\le s,\ \ \exists\boldsymbol{u}\in\boldsymbol{Z}^\lambda(\Gamma_{\boldsymbol{o}}),\ \boldsymbol{u}\neq 0,\ \chi_{\boldsymbol{o}}\boldsymbol{u}\in H_N(\Gamma_{\boldsymbol{o}})\Big\}.$$

Then instead of (3.6), we can deduce from Lemma 3.1

$$\begin{aligned}\Lambda^s_{(H)}(\Gamma)\ =\ &\Big\{\mu-1\ \big|\quad \mu\in(1,s+1)\cap\Lambda^{\mathrm{Dir}}(\Gamma)\Big\}\\ &\qquad\cup\ \ \Big\{\mu+1\ \big|\quad \mu\in(-1,s-1)\cap\Lambda^{\mathrm{Dir}}(\Gamma)\Big\}.\end{aligned}\tag{3.9}$$

Thus we see that both sets $\Lambda^s_{(X)}(\Gamma)$ and $\Lambda^s_{(H)}(\Gamma)$ are different if and only if $\Lambda^{\mathrm{Dir}}(\Gamma)$ has elements in the interval $[-1,+1]$, i.e. if $\omega>\pi$ (non-convex corner). Then

$$\Lambda^s_{(X)}=\Big\{\frac{\pi}{\omega}-1\Big\}\cup(\Lambda^s_{(X)}\cap\Lambda^s_{(H)})\quad\text{and}\quad\Lambda^s_{(H)}=\Big\{1-\frac{\pi}{\omega}\Big\}\cup(\Lambda^s_{(X)}\cap\Lambda^s_{(H)}).$$

4 Regularity and singularities on polyhedra (general theory)

As explained in the previous section, we continue our investigations of “model situations”, now in dimension 3. So we assume that $\Omega\subset\mathbb{R}^3$ is pseudo-polyhedral with ε and μ equal to the identity matrix. Thus the “electric” problem under consideration is:

$$\boldsymbol{u}\in X_N,\quad\forall\boldsymbol{v}\in X_N,\quad\int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}+\operatorname{div}\boldsymbol{u}\,\operatorname{div}\boldsymbol{v}=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v},\tag{4.1}$$

and the “magnetic” problem is the same with the variational space X_T instead X_N.

The corresponding boundary value problem (2.3) is

$$\begin{cases}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}-\mathbf{grad}\operatorname{div}\boldsymbol{u}=\boldsymbol{f} & \text{in }\ \Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=0\quad\text{and}\quad\operatorname{div}\boldsymbol{u}=0 & \text{on }\ \partial\Omega,\\ \boldsymbol{u}\in L^2(\Omega)^3,\quad\mathbf{curl}\,\boldsymbol{u}\in L^2(\Omega)^3\quad\text{and}\quad\operatorname{div}\boldsymbol{u}\in H^1(\Omega),\end{cases}\tag{4.2}$$

and the magnetic problem is the same with the boundary conditions $\boldsymbol{u}\cdot\boldsymbol{n}=0$ and $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=0$ on $\partial\Omega$ instead.

Of course, the interior operator $\mathbf{curl}\,\mathbf{curl}-\mathbf{grad}\,\mathrm{div}$ is the vector Laplace operator. But we prefer to keep the formulation via first order operators, because this formulation corresponds better to the weak formulation (4.1), which is also reflected in the boundary conditions.

We assume from now on that the right-hand side $\boldsymbol{f}$ belongs to $H^{s-1}(\Omega)^3$, with $s\geq 1$. As in dimension 2, the solutions $\boldsymbol{u}$ are regular in any neighborhood which does not meet any corner or edge of Ω. So the singular behavior of $\boldsymbol{u}$ is attached to corners and edges. In this section, we extract from KONDRAT'EV [15] and DAUGE [10] general information concerning the regularity and the main singularities of $\boldsymbol{u}$: these are determined by sets of *admissible exponents* Λ^s_N and Λ^s_T and their corresponding *admissible singularity* spaces $\boldsymbol{Z}^\lambda_N$ and $\boldsymbol{Z}^\lambda_T$, for each corner $\boldsymbol{c}$ and each edge $\boldsymbol{e}$. The admissible exponents can be seen as forming a spectrum and the admissible singularities as eigenvectors. The particular study of the above sets and spaces is postponed to sections 6 and 7.

4.a Corners and edges

Let $\mathscr{C}$ be the set of the corners $\boldsymbol{c}$ of Ω, that we define similarly to the corners of a polygon, with the requirement that for any $\boldsymbol{c}\in\mathscr{C}$ the corresponding cone $\Gamma_{x,i}$, that we denote $\Gamma_{\boldsymbol{c}}$, is a non-trivial cone (i.e. it is neither a half space nor a dihedron). The corresponding neighborhood $\Omega_{x,i}$ is denoted by $\mathscr{V}_{\boldsymbol{c}}$. In local spherical coordinates, the cone $\Gamma_{\boldsymbol{c}}$ is:

$$\Gamma_c=\{(\rho_c,\vartheta_c)\mid \rho_c>0,\ \vartheta_c\in G_c\subset\mathbb{S}^2\}\quad\text{with } G_c \text{ a spherical polygonal domain.}$$

Let $\mathscr{E}$ be the set of the (open) edges $\boldsymbol{e}$ of Ω: for each point $x\in\boldsymbol{e}$, there exists a neighborhood $\Omega_{x,i}$ in which Ω coincides with the wedge $\Gamma_{\boldsymbol{e}}\times\mathbb{R}$, where $\Gamma_{\boldsymbol{e}}$ is a plane sector given in local polar coordinates by

$$\Gamma_e=\{(r_e,\theta_e)\mid r_e>0,\ 0<\theta_e<\omega_e\}\quad\text{with } \omega_e \text{ the opening of } \Gamma_e\,.$$

We obtain local cylindrical coordinates by the adjunction of a coordinate z_e along the edge $\boldsymbol{e}$.

We are now going to introduce the "eigenspaces" of pseudo-homogeneous functions $Z^\lambda_N(\Gamma)$ (with $\lambda\in\mathbb{C}$ and Γ being alternatively $\Gamma_{\boldsymbol{c}}$ or $\Gamma_{\boldsymbol{e}}$) corresponding to the "electric" boundary conditions, the procedure for the "magnetic" conditions being strictly similar.

4.b Pseudo-homogeneous functions

If $\boldsymbol{c}$ is a corner, let $\boldsymbol{S}^\lambda_N(\Gamma_{\boldsymbol{c}})$ be the space

$$\boldsymbol{S}^\lambda_N(\Gamma_c)=\Big\{\boldsymbol{u}\in X^{\mathrm{loc}}_N(\Gamma^*_c)\mid\ \mathrm{div}\,\boldsymbol{u}\in H^1_{\mathrm{loc}}(\Gamma^*_c),\ \ \boldsymbol{u}(x)=\rho_c^\lambda\sum_{q=0}^{Q}\log^q\rho_c\,\boldsymbol{u}_q(\vartheta_c)\Big\},\qquad(4.3)$$

where $\boldsymbol{u} \in X_N^{\rm loc}(\Gamma_{\boldsymbol{c}}^*)$ and $\operatorname{div} \boldsymbol{u} \in H^1_{\rm loc}(\Gamma_{\boldsymbol{c}}^*)$ mean that for any bounded open set $\mathscr{O}$ such that $\boldsymbol{c} \notin \overline{\mathscr{O}}$, $\boldsymbol{u}$ belongs to $X_N(\Gamma_{\boldsymbol{c}} \cap \mathscr{O})$ and $\operatorname{div} \boldsymbol{u} \in H^1(\Gamma_{\boldsymbol{c}} \cap \mathscr{O})$. This contains two informations:

- $\boldsymbol{u} \times \boldsymbol{n}$ is zero on $\partial\Gamma_{\boldsymbol{c}}$,
- $\boldsymbol{u}$ belongs to $H(\mathbf{curl})$ and $\operatorname{div} \boldsymbol{u}$ belongs to H^1 near the edges of $\Gamma_{\boldsymbol{c}}$.

This definition is slightly more complicated than in [10] (where the variational spaces are classical subspaces of $H^m(\Omega)$), because we have to take into account the variational spaces and the basic regularity. The definition of $\boldsymbol{S}_T^\lambda(\Gamma_{\boldsymbol{c}})$ is similar.

We now define $\boldsymbol{Y}_N^\lambda(\Gamma_{\boldsymbol{c}})$ as the subspace of $\boldsymbol{S}_N^\lambda(\Gamma_{\boldsymbol{c}})$:

$$\begin{aligned}\boldsymbol{Y}_N^\lambda(\Gamma_{\boldsymbol{c}}) = \Big\{\boldsymbol{u} \in \boldsymbol{S}_N^\lambda(\Gamma_{\boldsymbol{c}}) \mid \quad & \operatorname{div} \boldsymbol{u} = 0 \ \text{ on } \ \partial\Gamma_{\boldsymbol{c}}, \\ & \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\operatorname{div} \boldsymbol{u} \ \text{ is polynomial}\Big\}.\end{aligned} \tag{4.4}$$

If λ is a positive integer, $\boldsymbol{Y}_N^\lambda(\Gamma_{\boldsymbol{c}})$ contains the space $\boldsymbol{P}_N^\lambda(\Gamma_{\boldsymbol{c}})$ of homogeneous polynomial fields $\boldsymbol{u}$ of degree λ satisfying the boundary conditions $\boldsymbol{u} \times \boldsymbol{n} = 0$ and $\operatorname{div} \boldsymbol{u} = 0$. That is why we introduce a complement $\boldsymbol{Z}_N^\lambda(\Gamma_{\boldsymbol{c}})$ of $\boldsymbol{P}_N^\lambda(\Gamma_{\boldsymbol{c}})$ in $\boldsymbol{Y}_N^\lambda(\Gamma_{\boldsymbol{c}})$

$$\boldsymbol{Y}_N^\lambda(\Gamma_{\boldsymbol{c}}) = \boldsymbol{Z}_N^\lambda(\Gamma_{\boldsymbol{c}}) \oplus \boldsymbol{P}_N^\lambda(\Gamma_{\boldsymbol{c}}). \tag{4.5}$$

Of course, if λ is not a positive integer, $\boldsymbol{Z}_N^\lambda(\Gamma_{\boldsymbol{c}})$ is simply

$$\Big\{\boldsymbol{u} \in \boldsymbol{S}_N^\lambda(\Gamma_{\boldsymbol{c}}) \mid \ \operatorname{div} \boldsymbol{u} = 0 \text{ on } \partial\Gamma_{\boldsymbol{c}}, \ \ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\operatorname{div} \boldsymbol{u} = 0\Big\}.$$

We denote by $\Lambda_N(\Gamma_{\boldsymbol{c}})$ the set of $\lambda \in \mathbb{C}$ such that $\boldsymbol{Z}_N^\lambda(\Gamma_{\boldsymbol{c}})$ is not reduced to $\{0\}$. This set can be understood as a spectrum. It is discrete and has the following property: in each strip $\operatorname{Re}\lambda \in [a,b]$, the number of elements of $\Lambda_N(\Gamma_{\boldsymbol{c}})$ is finite. Moreover the eigenspaces $\boldsymbol{Z}_N^\lambda(\Gamma_{\boldsymbol{c}})$ are finite-dimensional.

If $\boldsymbol{e}$ is an edge, the introduction of the corresponding spaces on Γ_e requires the elimination of the tangential derivatives in the operator $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\operatorname{div} \boldsymbol{u}$ and also in the natural boundary conditions. Let $\boldsymbol{y}_e$ be cartesian coordinates in the normal plane to the edge $\boldsymbol{e}$, and we denote by a subscript $_y$ the operators from which the tangential derivatives are eliminated. We can introduce $\boldsymbol{S}_N^\lambda(\Gamma_e)$ as the space

$$\boldsymbol{S}_N^\lambda(\Gamma_e) = \Big\{\boldsymbol{u} \in X_N^{\rm loc}(\Gamma_e^*) \mid \ \operatorname{div}_y \boldsymbol{u} \in H^1_{\rm loc}(\Gamma_e^*), \ \ \boldsymbol{u}(x) = r_e^\lambda \sum_{q=0}^{Q} \log^q r_e\, \boldsymbol{v}_q(\theta_e)\Big\}, \tag{4.6}$$

where $\boldsymbol{u} \in X_N^{\rm loc}(\Gamma_e^*)$ and $\operatorname{div}_y \boldsymbol{u} \in H^1_{\rm loc}(\Gamma_e^*)$ mean that for any bounded open set $\mathscr{O}$ such that $0 \notin \overline{\mathscr{O}}$, $\boldsymbol{u}$ and $\mathbf{curl}_y\, \boldsymbol{u}$ belong to $L^2(\Gamma_e \cap \mathscr{O})^3$ and $\operatorname{div}_y \boldsymbol{u} \in H^1(\Gamma_e \cap \mathscr{O})$. The definition of $\boldsymbol{S}_T^\lambda(\Gamma_e)$ is similar.

We now define $\boldsymbol{Y}_N^\lambda(\Gamma_e)$ as the subspace of $\boldsymbol{S}_N^\lambda(\Gamma_e)$:

$$\begin{aligned}\boldsymbol{Y}_N^\lambda(\Gamma_e) = \Big\{\boldsymbol{u} \in \boldsymbol{S}_N^\lambda(\Gamma_e) \mid \quad & \operatorname{div}_y \boldsymbol{u} = 0 \ \text{ on } \ \partial\Gamma_e, \\ & \mathbf{curl}_y\, \mathbf{curl}_y\, \boldsymbol{u} - \mathbf{grad}_y \operatorname{div}_y \boldsymbol{u} \ \text{ is polynomial}\Big\},\end{aligned} \tag{4.7}$$

and choose a complement $\boldsymbol{Z}_N^\lambda(\Gamma_e)$ of $\boldsymbol{P}_N^\lambda(\Gamma_e)$ in $\boldsymbol{Y}_N^\lambda(\Gamma_e)$:

$$\boldsymbol{Y}_N^\lambda(\Gamma_e) = \boldsymbol{Z}_N^\lambda(\Gamma_e) \oplus \boldsymbol{P}_N^\lambda(\Gamma_e). \tag{4.8}$$

We denote by $\Lambda_N(\Gamma_e)$ the set of $\lambda \in \mathbb{C}$ such that $\boldsymbol{Z}_N^\lambda(\Gamma_e)$ is not reduced to $\{0\}$. Like for corners $\boldsymbol{c}$, this set is discrete, $\boldsymbol{Z}_N^\lambda(\Gamma_e)$ is finite-dimensional and there holds: in each strip $\operatorname{Re}\lambda \in [a,b]$, the number of elements of $\Lambda_N(\Gamma_e)$ is finite.

4.c Regularity and singularities

Let σ be ≥ -1. We introduce the following subsets of $\Lambda_N(\Gamma_c)$ and $\Lambda_N(\Gamma_e)$ which correspond to *admissible singular functions* contained in the "energy space", *cf* (2.3d).

Let $\chi_c = \chi(\rho_c)$ be a smooth cut-off function with support in $\mathscr{V}_c$ and equal to 1 in a neighborhood of $\boldsymbol{c}$. We set

$$\Lambda_N^\sigma(\Gamma_c) = \Big\{\lambda \in \Lambda_N(\Gamma_c) \mid -\tfrac{3}{2} < \operatorname{Re}\lambda \leq \sigma - \tfrac{1}{2}, \\ \exists \boldsymbol{u} \in \boldsymbol{Z}_N^\lambda(\Gamma_c),\ \boldsymbol{u} \neq 0,\ \ \chi_c\boldsymbol{u} \in X_N(\Gamma_c),\ \operatorname{div}(\chi_c\boldsymbol{u}) \in H^1(\Gamma_c)\Big\}.$$

Similarly, with χ_e a smooth cut-off function equal to 1 in a neighborhood of 0:

$$\Lambda_N^\sigma(\Gamma_e) = \Big\{\lambda \in \Lambda_N(\Gamma_e) \mid \ \ -1 < \operatorname{Re}\lambda \leq \sigma, \\ \exists \boldsymbol{u} \in \boldsymbol{Z}_N^\lambda(\Gamma_e),\ \boldsymbol{u} \neq 0,\ \ \chi_e\boldsymbol{u} \in X_N(\Gamma_e),\ \operatorname{div}_y(\chi_e\boldsymbol{u}) \in H^1(\Gamma_e)\Big\}.$$

Note that the lower bound for $\operatorname{Re}\lambda$ is but a consequence of the condition $\chi\boldsymbol{u} \in L^2(\Gamma)^3$.

Here follows the regularity statement

Theorem 4.1 *Let $s \geq 1$ and $\sigma \in (-1, s]$. If*

- *for any $\boldsymbol{c} \in \mathscr{C}$, $\Lambda_N^\sigma(\Gamma_c)$ is empty,*
- *for any $\boldsymbol{e} \in \mathscr{E}$, $\Lambda_N^\sigma(\Gamma_e)$ is empty,*

then for any data $\boldsymbol{f} \in H^{s-1}(\Omega)^3$ the solution of problem (4.1) *belongs to $H^{\sigma+1}(\Omega)^3$.*

We are going to describe now the "first level" singularities, when the sets Λ_N^σ are not all empty. For any $\boldsymbol{c} \in \mathscr{C}$ and $\lambda \in \Lambda_N^\sigma(\Gamma_c)$ let $(\boldsymbol{U}_c^{\lambda,p})_p$ be a basis of $\boldsymbol{Z}_N^\lambda(\Gamma_c)$, and similarly for any $\boldsymbol{e} \in \mathscr{E}$ and $\lambda \in \Lambda_N^\sigma(\Gamma_e)$ let $(\boldsymbol{U}_e^{\lambda,p})_p$ be a basis of $\boldsymbol{Z}_N^\lambda(\Gamma_e)$. The corner singularities are simply a linear combination of the $\boldsymbol{U}_c^{\lambda,p}$, whereas the edge singularities are more difficult to describe:

We use a smooth function d_e on the closed edge $\bar{\boldsymbol{e}}$, which is equivalent to the distance to the ends of $\boldsymbol{e}$: if for example $\boldsymbol{e} = \{x \mid r_e = 0,\ z_e \in (-1,+1)\}$, we can take $d_e(z_e) = 1 - z_e^2$. We have now to introduce the special weighted Sobolev spaces $\mathbb{V}_\eta^m(\boldsymbol{e})$ which are the correct spaces for the singularity coefficients γ, defined for $m \in \mathbb{N}$ and $\eta \in \mathbb{R}$ by

$$\mathbb{V}_\eta^m(\boldsymbol{e}) = \Big\{\gamma \in L^2(\boldsymbol{e}) \mid \quad (d_e)^{\eta+k}\,\partial_{z_e}^k\gamma \in L^2(\boldsymbol{e}), \quad k = 0,1,\ldots,m\Big\}$$

and by interpolation for non-integers m. We also need a smoothing operator $\mathscr{K}[\cdot]$ which acts like a lifting of functions on $\boldsymbol{e}$ into Ω: in order to define it, we introduce the stretched variable

$$\tilde{z}_e = \int_0^{z_e} \frac{1}{d_e(z)}\, dz,$$

where $z = 0$ corresponds to an interior point of $\boldsymbol{e}$. The change of variable $z_e \mapsto \tilde{z}_e$ is one to one $\boldsymbol{e} \to \mathbb{R}$ and for any function γ defined on $\boldsymbol{e}$, we set $\tilde{\gamma}(\tilde{z}_e) = \gamma(z_e)$. Then $\mathscr{K}[\gamma](\rho_e, \theta_e, z_e)$ is the convolution operator with respect to $\tilde{z}_e$:

$$\mathscr{K}[\gamma](\rho_e, \theta_e, z_e) = \int_{\mathbb{R}} \frac{1}{\rho_e} \varphi\Big(\frac{t}{\rho_e}\Big)\, \tilde{\gamma}(t - \tilde{z}_e)\, dt \quad \text{with} \quad \rho_e = \frac{r_e}{d_e},$$

where φ is a smooth function in $\mathscr{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi = 1$.

We are now ready to give a statement of splitting in regular and singular parts: the assumptions *(i)* and *(ii)* below are the usual necessary ones, see [10, Thm 17.13], but the assumption *(iii)* only aims at simplifying the expression of the edge singularities.

Theorem 4.2 *Let* $s \geq 1$ *and* $\sigma \in [0, s]$. *If*

(i) For any $\boldsymbol{c} \in \mathscr{C}$, $\Lambda_N^\sigma(\Gamma_c)$ *does not meet the straight line* $\operatorname{Re} \lambda = \sigma - \frac{1}{2}$,

(ii) For any $\boldsymbol{e} \in \mathscr{E}$, $\Lambda_N^\sigma(\Gamma_e)$ *does not meet the straight line* $\operatorname{Re} \lambda = \sigma$,

(iii) For any $\boldsymbol{e} \in \mathscr{E}$, $\Lambda_N^\sigma(\Gamma_e)$ *is contained in a strip* $\operatorname{Re} \lambda \in (a, a+1)$, *and for any* $\lambda \in \Lambda_N^\sigma(\Gamma_e)$, $\boldsymbol{Z}_N^\lambda(\Gamma_e)$ *has no logarithmic element,*

then for any data $\boldsymbol{f} \in H^{s-1}(\Omega)^3$ *the solution* $\boldsymbol{u}$ *of problem* (4.1) *can be split in the following way*

$$\begin{aligned} \boldsymbol{u} = \boldsymbol{u}_{\mathrm{reg},\sigma} \quad &+ \sum_{c \in \mathscr{C}} \sum_{\lambda \in \Lambda_N^\sigma(\Gamma_c)} \sum_p \gamma_c^{\lambda,p}\, \chi_c(\rho_c)\, \boldsymbol{U}_c^{\lambda,p}(\rho_c, \vartheta_c) \\ &\qquad + \sum_{e \in \mathscr{E}} \sum_{\lambda \in \Lambda_N^\sigma(\Gamma_e)} \sum_p \mathscr{K}[\gamma_e^{\lambda,p}]\, \chi_e(\rho_e)\, \boldsymbol{U}_e^{\lambda,p}(\rho_e, \theta_e) \end{aligned} \tag{4.9}$$

with $\boldsymbol{u}_{\mathrm{reg},\sigma} \in H^{\sigma+1}(\Omega)^3$, $\gamma_c^{\lambda,p} \in \mathbb{C}$ *and the edge coefficients* $\gamma_e^{\lambda,p} \in \mathbb{V}_{-\sigma}^{s-\operatorname{Re}\lambda}(\boldsymbol{e})$.

Remark 4.3 1) In the usual statements of this sort, σ is taken equal to s. Here we allow intermediate splittings. Note that the influence of the regularity of the data subsists in the regularity of the edge coefficients.

2) If only the hypotheses *(i)* and *(ii)* hold, we still have an expansion of $\boldsymbol{u}$ in (optimal) regular and singular parts, but the edge contribution is still more complicated, due to possible Keldysh chains and supplementary "shadow" terms. ■

In section 8, we will give more explicit splittings, based on the knowledge of the sets of admissible exponents $\Lambda_N^\sigma(\Gamma_c)$ and $\Lambda_N^\sigma(\Gamma_e)$, and of the corresponding spaces of singular functions $\boldsymbol{Z}_N^\lambda(\Gamma_c)$ and $\boldsymbol{Z}_N^\lambda(\Gamma_e)$.

5 Maxwell edge singularities

Under the assumptions of the last section, our aim is to describe explicitly the spaces $\boldsymbol{Z}^\lambda$ (4.5) and (4.8) attached to each corner and edge of Ω. In this section, we treat the edges. As a byproduct, we will also obtain the singularities at the corners of a plane polygonal domain.

5.a 3D Maxwell singularities in a plane sector

The boundary value problem is then two-dimensional and involves a plane sector Γ of opening $\omega \in (0, 2\pi]$, $\omega \neq \pi$; the polar coordinates are denoted (r, θ), the cartesian coordinates in the plane of Γ are denoted $\boldsymbol{y}$, and z is a perpendicular coordinate.

Let $\lambda \in \mathbb{C}$. We search for non-polynomial solutions $\boldsymbol{u} \in \boldsymbol{S}_N^\lambda(\Gamma)$ of the system

$$\mathbf{curl}_y\, \mathbf{curl}_y\, \boldsymbol{u} - \mathbf{grad}_y\, \mathrm{div}_y\, \boldsymbol{u} = \boldsymbol{f} \ \text{ in } \Gamma, \quad \boldsymbol{f} \ \text{ polynomial,}$$

with the natural boundary condition $\mathrm{div}_y\, \boldsymbol{u} = 0$ on $\partial\Omega$, and where the field $\boldsymbol{u}$ has still 3 components, and $\boldsymbol{S}_N^\lambda(\Gamma)$ is the space (4.6) of pseudo-homogeneous fields of degree λ. Let us note that, *when λ is not a positive integer, the above problem reduces to find the non-zero solutions $\boldsymbol{u} \in \boldsymbol{S}_N^\lambda(\Gamma)$ of $\mathbf{curl}_y\, \mathbf{curl}_y\, \boldsymbol{u} - \mathbf{grad}_y\, \mathrm{div}_y\, \boldsymbol{u} = 0$ with the same boundary conditions.*

Since the boundary of Γ is smooth outside its vertex 0, the above boundary value problem is elliptic, and the data are smooth, we can introduce the space $S^\lambda(\Gamma)$:

$$S^\lambda(\Gamma) = \Big\{ u = r^\lambda \sum_{q=0}^{Q} \log^q r\, u_q(\theta) \ \big| \ u_q \in \mathscr{C}^\infty([0,\omega]) \Big\}, \tag{5.1}$$

and solve equivalently

$$\begin{cases} \mathbf{curl}_y\, \mathbf{curl}_y\, \boldsymbol{u} - \mathbf{grad}_y\, \mathrm{div}_y\, \boldsymbol{u} = \boldsymbol{f} & \text{in } \Gamma, \quad \boldsymbol{f} \text{ polynomial,} \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{and} \quad \mathrm{div}_y\, \boldsymbol{u} = 0 & \text{on } \partial\Gamma, \\ \boldsymbol{u} \in S^\lambda(\Gamma)^3. \end{cases} \tag{5.2}$$

Let now $(\boldsymbol{v}, w)$ be the decomposition of the field $\boldsymbol{u}$ in the system of cartesian coordinates $(\boldsymbol{y}, z)$. Then system (5.2) is split into 2 independent problems:

$$\begin{cases} \mathbf{curl}\, \mathrm{curl}\, \boldsymbol{v} - \mathbf{grad}\, \mathrm{div}\, \boldsymbol{v} = \boldsymbol{f} & \text{in } \Gamma, \quad \boldsymbol{f} \text{ polynomial,} \\ \boldsymbol{v} \times \boldsymbol{n} = 0 \quad \text{and} \quad \mathrm{div}\, \boldsymbol{v} = 0 & \text{on } \partial\Gamma, \\ \boldsymbol{v} \in S^\lambda(\Gamma)^2, \end{cases} \tag{5.3}$$

and

$$\begin{cases} -\Delta w = f & \text{in } \Gamma, \quad f \text{ polynomial,} \\ w = 0 & \text{on } \partial\Gamma, \\ w \in S^\lambda(\Gamma). \end{cases} \tag{5.4}$$

Indeed problem (5.3) is also the problem attached to two-dimensional Maxwell equations in a polygonal domain.

The situation for the Dirichlet problem (5.4) is well known: let

$$\zeta = re^{i\theta} = y_1 + iy_2.$$

- if $\omega \neq 2\pi$, the set $\Lambda^{\rm Dir}(\Gamma)$ of singular exponents is $\{\frac{k\pi}{\omega},\ k \in \mathbb{Z}^*\}$ and, for $\lambda \in \Lambda^{\rm Dir}(\Gamma)$, the singularity space $Z^\lambda_{\rm Dir}(\Gamma)$ is generated by $\Phi^\lambda_{\rm Dir} = \operatorname{Im}\zeta^\lambda$ if $\lambda \notin \mathbb{N}$ and by $\Phi^\lambda_{\rm Dir} = \operatorname{Im}\zeta^\lambda \log\zeta + \omega(y_2/\sin\omega)^\lambda$ if $\lambda \in \mathbb{N}$;
- if $\omega = 2\pi$, the set $\Lambda^{\rm Dir}(\Gamma)$ of singular exponents is $\{\frac{k}{2},\ k \text{ odd}\}$ and, for any $\lambda \in \Lambda^{\rm Dir}(\Gamma)$, $Z^\lambda_{\rm Dir}(\Gamma)$ is generated by $\operatorname{Im}\zeta^\lambda$.

Let us consider now the two-dimensional "Maxwell-type" problem (5.3). We introduce two auxiliary scalar variables

$$\psi = \operatorname{curl}\boldsymbol{v} \quad \text{and} \quad q = \operatorname{div}\boldsymbol{v}. \tag{5.5}$$

Taking the divergence of the first line of (5.3) yields equation (5.6a) below and the next equations (5.6b) and (5.6c) are straightforward

$$-\Delta q = \operatorname{div}\boldsymbol{f} \ \text{ in } \Gamma \qquad \text{and} \quad q = 0 \ \text{ on } \partial\Gamma \qquad \text{with} \quad q \in S^{\lambda-1}(\Gamma). \tag{5.6a}$$

$$\mathbf{curl}\,\psi = \mathbf{grad}\,q + \boldsymbol{f} \ \text{ in } \Gamma \qquad \text{with} \quad \psi \in S^{\lambda-1}(\Gamma). \tag{5.6b}$$

$$\operatorname{curl}\boldsymbol{v} = \psi,\ \ \operatorname{div}\boldsymbol{v} = q \ \text{ in } \Gamma \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n} = 0 \ \text{ on } \partial\Gamma \quad \text{with} \quad \boldsymbol{v} \in S^\lambda(\Gamma)^2. \tag{5.6c}$$

We easily see that the system of equations (5.6) is equivalent to (5.3).

5.b Non-integral exponents

In order to solve system (5.6), we begin with the simpler situation when λ *is not a positive integer.* Then the above system of equations reduces to

$$-\Delta q = 0 \ \text{ in } \Gamma \qquad \text{and} \quad q = 0 \ \text{ on } \partial\Gamma \qquad \text{with} \quad q \in S^{\lambda-1}(\Gamma). \tag{5.7a}$$

$$\mathbf{curl}\,\psi = \mathbf{grad}\,q \ \text{ in } \Gamma \qquad \text{with} \quad \psi \in S^{\lambda-1}(\Gamma). \tag{5.7b}$$

$$\operatorname{curl}\boldsymbol{v} = \psi,\ \ \operatorname{div}\boldsymbol{v} = q \ \text{ in } \Gamma \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n} = 0 \ \text{ on } \partial\Gamma \quad \text{with} \quad \boldsymbol{v} \in S^\lambda(\Gamma)^2. \tag{5.7c}$$

We can split the solutions of system (5.7) into three natural types:

Type 1 $q = 0$, $\psi = 0$ and $\boldsymbol{v}$ general non-zero solution of (5.7c).

Type 2 $q = 0$, ψ general non-zero solution of (5.7b) and $\boldsymbol{v}$ particular solution of (5.7c).

Type 3 q general non-zero solution of (5.7a), ψ particular solution of (5.7b) and $\boldsymbol{v}$ particular solution of (5.7c).

Let us study successively these three types.

Type 1.
Since $\operatorname{curl} \boldsymbol{v} = 0$ on the simply connected domain Γ, $\boldsymbol{v}$ is the gradient of a function Φ. Thus we have:

$$\begin{cases} v_r := \cos\theta\, v_1 + \sin\theta\, v_2 = \partial_r \Phi \\ v_\theta := -\sin\theta\, v_1 + \cos\theta\, v_2 = \frac{1}{r}\partial_\theta \Phi, \end{cases}$$

whence (we denote by $\tilde{v}$ the function $\tilde{v}(r,\theta) = v(y)$)

$$\tilde{\Phi}(r,\theta) - \tilde{\Phi}(1,0) = \int_1^r \tilde{v}_r(r',0)\, dr' \;+\; r\int_0^\theta \tilde{v}_\theta(r,\theta')\, d\theta', \tag{5.8}$$

which proves that since $\boldsymbol{v}$ belongs to $S^\lambda(\Gamma)^2$, Φ is the sum of a function in $S^{\lambda+1}(\Gamma)$ and a constant. Therefore, Φ can be found in $S^{\lambda+1}(\Gamma)$. Then (5.6c) is equivalent to

$$\Delta\Phi = 0 \;\text{ in } \Gamma \quad \text{and} \quad \Phi = 0 \;\text{ on } \partial\Gamma \quad \text{with} \quad \Phi \in S^{\lambda+1}(\Gamma). \tag{5.9}$$

Hence, $\lambda+1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ and Φ belongs to the space generated by $\operatorname{Im}\zeta^{\lambda+1}$.

Type 2.
We easily see that ψ is zero and a particular solution of (5.7c) is $\boldsymbol{v} = 0$.

Type 3.
From equation (5.7a), we obtain that $\lambda-1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ and that q belongs to $Z_{\mathrm{Dir}}^{\lambda-1}(\Gamma)$: thus q is proportional to $\operatorname{Im}\zeta^{\lambda-1}$. Then it is easy to see that $\psi = -\operatorname{Re}\zeta^{\lambda-1}$ is a particular solution of (5.7b), and that $\boldsymbol{v} = \frac{1}{2\lambda}(\operatorname{Im}\zeta^\lambda, -\operatorname{Re}\zeta^\lambda)$ is a particular solution of (5.7c).

5.c Integral exponents

When λ is a positive integer, we are searching for non-polynomial solutions of system (5.6). Similarly to the case when λ is not an integer, we split the solutions of the system (5.6) into the three types:

Type 1 q and ψ are polynomial and $\boldsymbol{v}$ is a non-polynomial solution of (5.6c).

Type 2 q is polynomial, ψ is a non-polynomial solution of (5.6b) and $\boldsymbol{v}$ a particular solution of (5.6c).

Type 3 q is a non-polynomial solution of (5.6a), ψ a particular solution of (5.6b) and $\boldsymbol{v}$ a particular solution of (5.6c).

Now the arguments are based on the evaluation of dimensions of polynomial spaces. Let Q^λ be the space of homogeneous polynomials of degree λ, $P^\lambda(\Gamma)$ the subspace of $q \in Q^\lambda$ with zero traces on $\partial\Gamma$. We divide our study into three subcases:

(i) $\omega \neq 2\pi$ and $\lambda-1$ does not belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$: In equation (5.6a) the r.h.s. $\operatorname{div}\boldsymbol{f}$ is any polynomial in $Q^{\lambda-3}$, thus the dimension of the range is $(\lambda-2)_+$. The dimension of $P^{\lambda-1}(\Gamma)$ is $(\lambda-2)_+$ too. Moreover equation (5.6a) defines an operator from $P^{\lambda-1}(\Gamma)$

into $Q^{\lambda-3}$ which is one to one due to the assumption that $\lambda-1$ does not belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$. Therefore this operator is onto.

- The r.h.s. of equation (5.6b) is any field in $Q^{\lambda-2}\times Q^{\lambda-2}$ which is divergence free. Thus the dimension of its range is $2(\lambda-1)-(\lambda-2)_+$ which is equal to λ if $\lambda\ge 2$ and 0 if $\lambda=1$. The dimension of $Q^{\lambda-1}$ is equal to λ and equation (5.6b) defines an operator from $Q^{\lambda-1}$ into $\{\boldsymbol{g}\in Q^{\lambda-2}\times Q^{\lambda-2},\ \operatorname{div}\boldsymbol{g}=0\}$, which is one to one for any $\lambda\ge 2$, thus onto.
- The r.h.s. (ψ,q) of (5.6c) is any element of $Q^{\lambda-1}\times P^{\lambda-1}(\Gamma)$. Thus the dimension of its range is $\lambda+(\lambda-2)_+=2(\lambda-1)$ if $\lambda\ge 2$ and 1 if $\lambda=1$. The space of polynomial solutions of (5.7c) is

$$\{\boldsymbol{v}\in Q^{\lambda}\times Q^{\lambda}\mid\quad \boldsymbol{v}\times\boldsymbol{n}=0\quad\text{and}\quad \operatorname{div}\boldsymbol{v}=0\quad\text{on}\quad\partial\Gamma\}. \tag{5.10}$$

Its dimension is $2(\lambda+1)-4=2(\lambda-1)$ if $\lambda\ge 2$; if $\lambda=1$, its dimension is either 2 if $\cos\omega=0$ or 1 if not. If $\lambda+1$ does not belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$, we check that in any case the operator of equation (5.6c) is one to one, thus it is onto: the system (5.6) has only polynomial solutions. If $\lambda+1\in\Lambda^{\mathrm{Dir}}(\Gamma)$, its kernel is one-dimensional, and for $\lambda\ge 2$ we add to the above polynomial space (5.10) a singular function equal to the sum of $\mathbf{grad}(\operatorname{Im}\zeta^{\lambda+1}\log\zeta)$ and of a polynomial: we have found now a solution of type 1. For $\lambda=1$ finally, $\lambda+1\in\Lambda^{\mathrm{Dir}}(\Gamma)$ only if $\cos\omega=0$ and the operator of equation (5.6c) is onto and there is no singularity.

(ii) $\omega=2\pi$: The arguments are similar. Here $\lambda-1$ and $\lambda+1$ never belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$. The dimensions of the polynomial spaces involving boundary conditions are slightly different: $\dim P^{\lambda-1}(\Gamma)=\lambda-1$ and the operator of equation (5.6a) has a one-dimensional kernel generated by $\operatorname{Im}\zeta^{\lambda-1}$. Thus it is still onto from $P^{\lambda-1}(\Gamma)\to Q^{\lambda-3}$.

- The situation for equation (5.6b) is unchanged.
- The dimension of the space $Q^{\lambda-1}\times P^{\lambda-1}(\Gamma)$ is $2\lambda-1$ and the dimension of the space (5.10) is $2(\lambda+2)-2=2\lambda$. The kernel of the operator of equation (5.6c) is generated by $\mathbf{grad}\operatorname{Im}\zeta^{\lambda-1}$. Thus we have only polynomial solutions.

(iii) $\omega\ne 2\pi$ and $\lambda-1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ *:* Then the operator of equation (5.6a) has a one-dimensional kernel generated by $\operatorname{Im}\zeta^{\lambda-1}$ and it is onto from the space generated by the sum of $P^{\lambda-1}(\Gamma)$ and of $\Phi^{\lambda-1}_{\mathrm{Dir}}$ which is the sum of $\operatorname{Im}\zeta^{\lambda-1}\log\zeta$ and of a polynomial.

- Corresponding to this new solution q, we find a new solution of equation (5.6b) $\psi=\operatorname{Re}\zeta^{\lambda-1}\log\zeta$.
- Accordingly, we find a new solution $\boldsymbol{v}$ of equation (5.6c) in the form $\boldsymbol{v}=\frac{1}{2\lambda}(\operatorname{Im}\zeta^{\lambda}\log\zeta,-\operatorname{Re}\zeta^{\lambda}\log\zeta)$, which is a non-polynomial solution of type 3.

5.d Edge singularities

We summarize the results that we have just proved:

Lemma 5.1 *The set of the exponents* $\Lambda_N(\Gamma)$ *attached to a plane sector of opening* ω *is*

$$\Lambda_N(\Gamma) = \{\lambda \in \mathbb{R} \setminus \{1\} \;\mid\; \lambda - 1,\ \lambda \ \textit{ or } \lambda + 1 \textit{ belongs to } \Lambda^{\mathrm{Dir}}(\Gamma)\}$$

where $\Lambda^{\mathrm{Dir}}(\Gamma)$ *is the set of the Dirichlet exponents on* Γ *:*

$$\Lambda^{\mathrm{Dir}}(\Gamma) = \{\tfrac{k\pi}{\omega},\ k \in \mathbb{Z}^*\} \ \textit{ if } \ \omega \neq 2\pi \quad \textit{and} \quad \Lambda^{\mathrm{Dir}}(\Gamma) = \{\tfrac{k}{2},\ k \textit{ odd}\} \ \textit{ if } \ \omega = 2\pi.$$

The corresponding spaces $\boldsymbol{Z}_N^\lambda(\Gamma)$ *of singular functions* $\boldsymbol{u} = (\boldsymbol{v}, w)$ *are generated by:*

(i) If $\lambda + 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$ *, then* $\boldsymbol{v}$ *is a Maxwell singularity of type 1:*
If $\lambda \notin \mathbb{N}$ *,* $\boldsymbol{v} = \mathbf{grad}\,\mathrm{Im}\,\zeta^{\lambda+1}$ *, in other words*

$$(\boldsymbol{v}, w) = (r^\lambda \sin\lambda\theta,\ r^\lambda \cos\lambda\theta,\ 0).$$

If $\lambda \in \mathbb{N}$ *:* $\boldsymbol{v} = \mathbf{grad}\,\mathrm{Im}\,\zeta^{\lambda+1}\log\zeta$ *, in other words*

$$(\boldsymbol{v}, w) = (r^\lambda(\log r \sin\lambda\theta + \theta\cos\theta),\ r^\lambda(\log r\cos\lambda\theta - \theta\sin\theta),\ 0).$$

(ii) If $\lambda \in \Lambda^{\mathrm{Dir}}(\Gamma)$ *,* w *is a Dirichlet singularity:*

$$(\boldsymbol{v}, w) = (\mathbf{0}, r^\lambda \sin\lambda\theta) \ \textit{ if } \ \lambda \notin \mathbb{N} \quad \textit{and} \quad (\mathbf{0}, r^\lambda(\log r\sin\lambda\theta + \theta\cos\theta)) \ \textit{ if } \ \lambda \in \mathbb{N}\,.$$

(iii) If $\lambda - 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$ *,* $\boldsymbol{v}$ *is a Maxwell singularity of type 3:*
If $\lambda \notin \mathbb{N}$ *:*

$$(\boldsymbol{v}, w) = (r^\lambda \sin\lambda\theta,\ -r^\lambda\cos\lambda\theta,\ 0).$$

If $\lambda \in \mathbb{N}$ *:*

$$(\boldsymbol{v}, w) = (r^\lambda(\log r\sin\lambda\theta + \theta\cos\theta),\ -r^\lambda(\log r\cos\lambda\theta - \theta\sin\theta),\ 0).$$

According to the general theory stated in §4.c, we have to consider the subset of $\Lambda_N(\Gamma)$ satisfying (with χ a cut-off function which is equal to 1 in a neighborhood of the corner of Γ):

$$-1 < \lambda \quad \text{and} \quad \exists \boldsymbol{u} \in \boldsymbol{Z}_N^\lambda(\Gamma),\ \boldsymbol{u} \neq 0 \quad \chi\boldsymbol{u} \in X_N(\Gamma),\ \ \mathrm{div}_y(\chi\boldsymbol{u}) \in H^1(\Gamma). \tag{5.11}$$

We examine the effect of condition (5.11) on each of the singularities *(i)*, *(ii)* and *(iii)* in Lemma 5.1.

(i) In this case $\lambda = \frac{k\pi}{\omega} - 1$: as $k \neq 0$, the lowest possible value of k is 1. For the corresponding singularities $\boldsymbol{u} = (\boldsymbol{v}, 0)$ with $\boldsymbol{v}$ Maxwell singularity of type 1, the curl and the divergence are 0 (or polynomial if $\lambda \in \mathbb{N}$). Thus for any $k \geq 1$ such that $\frac{k\pi}{\omega}$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$, $\lambda = \frac{k\pi}{\omega} - 1$ satisfies (5.11).

(ii) If $\lambda = \frac{k\pi}{\omega}$, the conditions about the curl and the divergence of $\boldsymbol{u} = (\mathbf{0}, w)$ mean exactly that χw belongs to $H^1(\Gamma)$. Thus λ is > 0, and for any $k \geq 1$ such that $\frac{k\pi}{\omega}$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$, $\lambda = \frac{k\pi}{\omega}$ satisfies (5.11).

(iii) If $\lambda = \frac{k\pi}{\omega} + 1$, $\boldsymbol{u} = (\boldsymbol{v}, 0)$ with $\boldsymbol{v}$ a Maxwell singularity of type 3: the divergence of $\boldsymbol{v}$ is itself a non-regular pseudo-homogeneous function of degree $\lambda - 2$. Since $\operatorname{div}_y(\chi \boldsymbol{u}) \in H^1(\Gamma)$, we must have $\lambda - 2 > 0$. Then the other conditions in (5.11) are implied, and for any $k \geq 1$ such that $\frac{k\pi}{\omega} > 1$ and $\frac{k\pi}{\omega} \in \Lambda^{\mathrm{Dir}}(\Gamma)$, $\lambda = \frac{k\pi}{\omega} + 1$ satisfies (5.11).

Thus, for any value of the opening ω, the most singular among the admissible elements of the $\boldsymbol{Z}_N^\lambda(\Gamma)$ (*admissible singular functions*) is

$$\boldsymbol{u} = (\mathbf{grad}\, \operatorname{Im} \zeta^{\pi/\omega}, 0) \quad \text{corresponding to} \quad \lambda = \tfrac{\pi}{\omega} - 1.$$

If $\omega > \pi$, this is the only admissible singular function such that $\chi \boldsymbol{u}$ does not belong to $H^1(\Gamma)$.

6 Basic Laplace corner singularities

In order to continue the investigation of the Maxwell singularity spaces $\boldsymbol{Z}_N^\lambda(\Gamma)$ and $\boldsymbol{Z}_T^\lambda(\Gamma)$ when Γ is a three-dimensional polyhedral cone, in a preparatory step we recall well-known facts about the *basic* Laplace singularities for Dirichlet and Neumann boundary conditions. Here we mean that we do not investigate polynomials (either as right hands sides, nor as solutions of our problems in the pseudo-homogeneous function spaces).

The polar coordinates in the polyhedral cone are denoted (ρ, ϑ) and

$$\Gamma = \{(\rho, \vartheta) \mid \rho > 0,\ \vartheta \in G \subset \mathbb{S}^2\} \quad \text{with } G \text{ a spherical polygonal domain.}$$

In contrast with the two-dimensional situation when G is an interval (thus a smooth domain), we have to introduce spaces of pseudo-homogeneous functions where the regularity of the angular functions is fixed: for $\tau \in \mathbb{R}$

$$S_\tau^\lambda(\Gamma) = \Big\{ u = r^\lambda \sum_{q=0}^{Q} \log^q r\, u_q(\theta) \ \big|\ u_q \in H^\tau(G) \Big\}. \tag{6.1}$$

Of course, the space of pseudo-homogeneous functions adapted to the variational regularity of the Laplace operator is $S_1^\lambda(\Gamma)$. The set $\Lambda^{\mathrm{Dir}}(\Gamma)$ of the singular exponents of the Dirichlet problem is the set of the $\lambda \in \mathbb{C}$ such that there exist non-zero solutions to

$$\begin{cases} -\Delta u = 0 & \text{in } \Gamma, \\ u = 0 & \text{on } \partial\Gamma, \\ u \in S_1^\lambda(\Gamma), \end{cases} \tag{6.2}$$

and the space $Z_{\mathrm{Dir}}^\lambda(\Gamma)$ is the solution space. Similarly the set $\Lambda^{\mathrm{Neu}}(\Gamma)$ of the Neumann singular exponents is the set of the $\lambda \in \mathbb{C}$ such that there exist non-zero solutions to

$$\begin{cases} -\Delta u = 0 & \text{in } \Gamma, \\ \partial_n u = 0 & \text{on } \partial\Gamma, \\ u \in S_1^\lambda(\Gamma). \end{cases} \tag{6.3}$$

Let $\Delta_G^{\rm Dir}$ be the positive Laplace-Beltrami operator with Dirichlet conditions on G. The operator $\Delta_G^{\rm Dir}$ is self-adjoint with a compact inverse. Let $\mu_1^{\rm Dir} < \mu_2^{\rm Dir} \le \cdots$ be its spectrum and $\varphi_j^{\rm Dir}$ be corresponding eigenfunctions. Similarly $\Delta_G^{\rm Neu}$ denotes the Neumann Laplace-Beltrami operator on G, $\mu_0^{\rm Neu} = 0 < \mu_1^{\rm Neu} \le \mu_2^{\rm Neu} \le \cdots$ is its spectrum and $\varphi_j^{\rm Neu}$ are corresponding eigenfunctions. From the expression of Δ in polar coordinates

$$\Delta = \tfrac{1}{\rho^2}\Big((\rho\partial_\rho)^2 + \rho\partial_\rho - \Delta_G\Big)$$

we obtain that the sets of exponents $\Lambda^{\rm Dir}(\Gamma)$ and $\Lambda^{\rm Neu}(\Gamma)$ are given by the roots of the equations $\lambda(\lambda+1) = \mu_j$:

$$\Lambda^{\rm Dir}(\Gamma) = \Big\{-\tfrac12 \pm \sqrt{\mu_j^{\rm Dir} + \tfrac14},\ j \ge 1\Big\} \quad \Lambda^{\rm Neu}(\Gamma) = \Big\{-\tfrac12 \pm \sqrt{\mu_j^{\rm Neu} + \tfrac14},\ j \ge 0\Big\} \tag{6.4}$$

and the corresponding singularity spaces by:

$$Z^\lambda_{\rm Dir}(\Gamma) = \operatorname{span}\Big\{\rho^\lambda\varphi_j^{\rm Dir}(\vartheta)\ \big|\ \lambda = -\tfrac12 \pm \sqrt{\mu_j^{\rm Dir} + \tfrac14}\Big\}, \tag{6.5a}$$

$$Z^\lambda_{\rm Neu}(\Gamma) = \operatorname{span}\Big\{\rho^\lambda\varphi_j^{\rm Neu}(\vartheta)\ \big|\ \lambda = -\tfrac12 \pm \sqrt{\mu_j^{\rm Neu} + \tfrac14}\Big\}. \tag{6.5b}$$

7 Maxwell corner singularities

We continue the investigation of the Maxwell singularity spaces $\boldsymbol{Z}_N^\lambda$ and $\boldsymbol{Z}_T^\lambda$. After those attached to plane sectors corresponding to the edges of our polyhedral domain (or to the corners of a polygonal domain), we study in this section the spaces attached to Γ where Γ is a three-dimensional polyhedral cone. We recall that the polar coordinates are denoted by (ρ,ϑ) and $\Gamma = \{(\rho,\vartheta)\mid \rho>0,\ \vartheta\in G\subset\mathbb{S}^2\}$.

According to definitions (4.3)-(4.5) we have to solve Maxwell systems in the spaces of pseudo-homogeneous functions

$$\boldsymbol{S}_N^\lambda(\Gamma) = \Big\{\boldsymbol{u}\in X_N^{\rm loc}(\Gamma^*)\ |\ \operatorname{div}\boldsymbol{u}\in H^1_{\rm loc}(\Gamma^*),\ \ \boldsymbol{u}(x) = \rho^\lambda\sum_{q=0}^{Q}\log^q\!\rho\ \boldsymbol{v}_q(\vartheta)\Big\},$$

and

$$\boldsymbol{S}_T^\lambda(\Gamma) = \Big\{\boldsymbol{u}\in X_T^{\rm loc}(\Gamma^*)\ |\ \operatorname{div}\boldsymbol{u}\in H^1_{\rm loc}(\Gamma^*),\ \ \boldsymbol{u}(x) = \rho^\lambda\sum_{q=0}^{Q}\log^q\!\rho\ \boldsymbol{v}_q(\vartheta)\Big\},$$

i.e. to find the non-polynomial solutions of the system

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\operatorname{div}\boldsymbol{u} = \boldsymbol{f}\ \text{ in }\ \Gamma,\quad \boldsymbol{f}\ \text{ polynomial},$$

with either $\boldsymbol{u}\in\boldsymbol{S}_N^\lambda(\Gamma)$ and $\operatorname{div}\boldsymbol{u}=0$ on $\partial\Omega$ (electric), or $\boldsymbol{u}\in\boldsymbol{S}_T^\lambda(\Gamma)$ and $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = 0$ on $\partial\Omega$ (magnetic).

7.a Splitting of the problem

Here we concentrate first on the case when λ *is not a positive integer.* Thus, in the electric case, the problem reduces to finding non-zero solutions to

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Gamma, \\ \boldsymbol{u}\times\boldsymbol{n} = 0, \quad \mathrm{div}\,\boldsymbol{u} = 0 & \text{on } \partial\Gamma, \\ \boldsymbol{u}\in \boldsymbol{S}_N^\lambda(\Gamma). \end{cases} \tag{7.1}$$

Like in the case of plane sectors, we introduce the auxiliary unknowns

$$\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{u} \quad \text{and} \quad q = \mathrm{div}\,\boldsymbol{u}.$$

In contrast to the two-dimensional case, it is impossible to require the same type of regularity for $\boldsymbol{u}$, $\boldsymbol{\psi}$ and q on G, because of the edge singularities which appear as singularities at the corners of G. From the condition $\mathrm{div}\,\boldsymbol{u}\in H^1_{\mathrm{loc}}(\Gamma^*)$ contained in the definition of $\boldsymbol{S}_N^\lambda(\Gamma)$, we are lead to use the space $S_1^\lambda(\Gamma)$ adapted to $q=\mathrm{div}\,\boldsymbol{u}$, see (6.1). Concerning the field $\boldsymbol{\psi}$, we remark that the equality $\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{u}$ implies that $\mathrm{div}\,\boldsymbol{\psi}=0$ and since $\boldsymbol{u}\times\boldsymbol{n}=0$ on $\partial\Gamma$, then $\boldsymbol{\psi}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. Moreover, from the equality $\mathbf{curl}\,\boldsymbol{\psi} = \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u}$, and from the H^1 regularity of $\mathrm{div}\,\boldsymbol{u}$, we obtain that $\mathbf{curl}\,\boldsymbol{\psi}$ belongs to $L^2_{\mathrm{loc}}(\Gamma^*)$. Thus the natural space for $\boldsymbol{\psi}$ is $\boldsymbol{S}_T^\lambda(\Gamma)$ and it is now clear that problem (7.1) is equivalent to find non-zero solutions to the system

$$-\Delta q = 0 \ \text{ in } \Gamma \quad \text{and} \quad q=0 \ \text{ on } \partial\Gamma \qquad \text{with} \quad q\in S_1^{\lambda-1}(\Gamma). \tag{7.2a}$$

$$\mathbf{curl}\,\boldsymbol{\psi} = \mathbf{grad}\,q \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi}=0 \ \text{ in } \Gamma \qquad \text{with} \quad \boldsymbol{\psi}\in \boldsymbol{S}_T^{\lambda-1}(\Gamma). \tag{7.2b}$$

$$\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}=q \ \text{ in } \Gamma \qquad \text{with} \quad \boldsymbol{u}\in \boldsymbol{S}_N^{\lambda}(\Gamma). \tag{7.2c}$$

Now we see that the "electric" and "magnetic" boundary conditions appear simultaneously inside (7.2). Thus we have better to treat both conditions together. The "magnetic" problem corresponding to (7.1) is to find non-zero solutions to

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Gamma, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = 0 & \text{on } \partial\Gamma, \\ \boldsymbol{u}\in \boldsymbol{S}_T^\lambda(\Gamma), \end{cases} \tag{7.3}$$

and it is equivalent to find non-zero solutions to the system of three problems

$$-\Delta q = 0 \ \text{ in } \Gamma \quad \text{and} \quad \partial_n q=0 \ \text{ on } \partial\Gamma \qquad \text{with} \quad q\in S_1^{\lambda-1}(\Gamma). \tag{7.4a}$$

$$\mathbf{curl}\,\boldsymbol{\psi} = \mathbf{grad}\,q \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi}=0 \ \text{ in } \Gamma \qquad \text{with} \quad \boldsymbol{\psi}\in \boldsymbol{S}_N^{\lambda-1}(\Gamma). \tag{7.4b}$$

$$\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}=q \ \text{ in } \Gamma \qquad \text{with} \quad \boldsymbol{u}\in \boldsymbol{S}_T^{\lambda}(\Gamma). \tag{7.4c}$$

Like for the plane sectors, the solutions of systems (7.2) and (7.4) belong to one of three types:

Type 1 $q = 0$, $\boldsymbol{\psi} = 0$ and $\boldsymbol{u}$ general non-zero solution of (7.2c), resp. (7.4c).

Type 2 $q = 0$, $\boldsymbol{\psi}$ general non-zero solution of (7.2b), resp. (7.4b) and $\boldsymbol{u}$ particular solution of (7.2c), resp. (7.4c).

Type 3 q general non-zero solution of (7.2a), resp. (7.4a), $\boldsymbol{\psi}$ particular solution of (7.2b), resp. (7.4b) and $\boldsymbol{u}$ particular solution of (7.2c), resp. (7.4c).

7.b Explicit solutions of first order problems

In the case of plane sectors, we have exhibited the singular functions (Lemma 5.1). Here, on three-dimensional cones, the Laplace singularities have no more explicit description than (6.5), but, relying on them, we are able to provide completely explicitly the three types of Maxwell singularities. This section is devoted to the description of solution formulas for the first order problems (7.2) and (7.4). All these formulas are based on the scalar product or the vector product with the vector $\boldsymbol{x}$, with $\boldsymbol{x}$ denoting the vector of cartesian coordinates (x_1, x_2, x_3).

We begin with three series of formulas. First we give product laws: $\boldsymbol{a}$ and $\boldsymbol{b}$ denoting vector fields and γ being a scalar function on $\mathbb{R}^3$, we have

$$\begin{aligned}
\mathbf{grad}(\boldsymbol{a}\cdot\boldsymbol{b}) &= (\boldsymbol{a}\cdot\mathbf{grad})\,\boldsymbol{b} + (\boldsymbol{b}\cdot\mathbf{grad})\,\boldsymbol{a} + \boldsymbol{a}\times\mathbf{curl}\,\boldsymbol{b} + \boldsymbol{b}\times\mathbf{curl}\,\boldsymbol{a}, && (7.5a)\\
\mathbf{curl}(\boldsymbol{a}\times\boldsymbol{b}) &= (\boldsymbol{b}\cdot\mathbf{grad})\,\boldsymbol{a} - (\boldsymbol{a}\cdot\mathbf{grad})\,\boldsymbol{b} + \boldsymbol{a}\,\operatorname{div}\boldsymbol{b} - \boldsymbol{b}\,\operatorname{div}\boldsymbol{a}, && (7.5b)\\
\operatorname{div}(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{b}\cdot\mathbf{curl}\,\boldsymbol{a} - \boldsymbol{a}\cdot\mathbf{curl}\,\boldsymbol{b}, && (7.5c)\\
\mathbf{curl}(\gamma\boldsymbol{a}) &= \gamma\,\mathbf{curl}\,\boldsymbol{a} + \mathbf{grad}\,\gamma\times\boldsymbol{a}, && (7.5d)\\
\operatorname{div}(\gamma\boldsymbol{a}) &= \gamma\operatorname{div}\boldsymbol{a} + \mathbf{grad}\,\gamma\cdot\boldsymbol{a}. && (7.5e)
\end{aligned}$$

Now, using the above formulas for the field $\boldsymbol{x}$ which satisfies

$$\operatorname{div}\boldsymbol{x} = 3, \quad \mathbf{curl}\,\boldsymbol{x} = 0, \quad \boldsymbol{x}\cdot\mathbf{grad} = \rho\partial_\rho \quad \text{and} \quad \mathbf{grad}\,\boldsymbol{x} = \mathrm{I},$$

we obtain for any field $\boldsymbol{a}$ and scalar q

$$\begin{aligned}
\mathbf{grad}(\boldsymbol{a}\cdot\boldsymbol{x}) &= (\rho\partial_\rho + 1)\boldsymbol{a} + \boldsymbol{x}\times\mathbf{curl}\,\boldsymbol{a}, && (7.6a)\\
\mathbf{curl}(\boldsymbol{a}\times\boldsymbol{x}) &= (\rho\partial_\rho + 2)\boldsymbol{a} - \boldsymbol{x}\,\operatorname{div}\boldsymbol{a}, && (7.6b)\\
\operatorname{div}(\boldsymbol{a}\times\boldsymbol{x}) &= \boldsymbol{x}\cdot\mathbf{curl}\,\boldsymbol{a}, && (7.6c)\\
\mathbf{curl}(q\boldsymbol{x}) &= \mathbf{grad}\,q\times\boldsymbol{x}, && (7.6d)\\
\operatorname{div}(q\boldsymbol{x}) &= (\rho\partial_\rho + 3)q. && (7.6e)
\end{aligned}$$

Finally, with $\gamma = \rho^2$ and $\boldsymbol{a} = \mathbf{grad}\,q$, (7.5d) and (7.5e) yield

$$\begin{aligned}
\mathbf{curl}(\rho^2\,\mathbf{grad}\,q) &= -2\,\mathbf{grad}\,q\times\boldsymbol{x}, && (7.6f)\\
\operatorname{div}(\rho^2\,\mathbf{grad}\,q) &= 2\rho\partial_\rho q + \rho^2\Delta q. && (7.6g)
\end{aligned}$$

These formulas allow to solve first order problems in the subspaces of homogeneous elements of our pseudo-homogeneous spaces S_1^λ, $\boldsymbol{S}_N^\lambda$ and $\boldsymbol{S}_T^\lambda$:

$$\mathring{S}_1^\lambda(\Gamma) = \Big\{ u = \rho^\lambda \upsilon(\vartheta) \mid \ \upsilon \in H^1(G) \Big\}, \tag{7.7}$$

$$\mathring{\boldsymbol{S}}_N^\lambda(\Gamma) = \Big\{ \boldsymbol{u} \in X_N^{\rm loc}(\Gamma^*) \mid \ \operatorname{div} \boldsymbol{u} \in H^1_{\rm loc}(\Gamma^*), \ \ \boldsymbol{u}(x) = \rho^\lambda \boldsymbol{\upsilon}(\vartheta) \Big\}, \tag{7.8}$$

and

$$\mathring{\boldsymbol{S}}_T^\lambda(\Gamma) = \Big\{ \boldsymbol{u} \in X_T^{\rm loc}(\Gamma^*) \mid \ \operatorname{div} \boldsymbol{u} \in H^1_{\rm loc}(\Gamma^*), \ \ \boldsymbol{u}(x) = \rho^\lambda \boldsymbol{\upsilon}(\vartheta) \Big\}. \tag{7.9}$$

As an easy consequence of formula (7.6a), we can solve the equation $\mathbf{grad}\, \Phi = \boldsymbol{u}$ with Dirichlet or Neumann boundary conditions:

Lemma 7.1 *Let $\boldsymbol{u}$ belong to $\boldsymbol{S}_N^\lambda(\Gamma)$ or $\boldsymbol{S}_T^\lambda(\Gamma)$.*
(i) *Then $\boldsymbol{u} \cdot \boldsymbol{x}$ belongs to $S_1^{\lambda+1}(\Gamma)$.*
(ii) *We assume that $\lambda \neq -1$ and that $\mathbf{curl}\, \boldsymbol{u} = 0$. If moreover $\boldsymbol{u}$ is homogeneous, i.e. $\boldsymbol{u} \in \mathring{\boldsymbol{S}}_N^\lambda(\Gamma)$, resp. $\mathring{\boldsymbol{S}}_T^\lambda(\Gamma)$, then Φ defined as*

$$\Phi = \frac{\boldsymbol{u} \cdot \boldsymbol{x}}{\lambda + 1} \quad \in \quad \mathring{S}_1^{\lambda+1}(\Gamma), \tag{7.10}$$

solves the equation $\mathbf{grad}\, \Phi = \boldsymbol{u}$, with zero Dirichlet, resp. Neumann boundary conditions on $\partial\Gamma$.

Proof. *(i)* A first consequence of formula (7.6a) is that $\mathbf{grad}(\boldsymbol{u} \cdot \boldsymbol{x})$ belongs to $S_0^\lambda(\Gamma)^3$, thus $\boldsymbol{u} \cdot \boldsymbol{x}$ has the correct regularity outside the corner of Γ.

(ii) As an obvious consequence of the fact that if $\boldsymbol{u}$ belongs to $\mathring{\boldsymbol{S}}^\lambda(\Gamma)$, then $\rho \partial_\rho \boldsymbol{u} = \lambda \boldsymbol{u}$, we obtain that $\mathbf{grad}\, \Phi = \boldsymbol{u}$. Moreover, if $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Gamma$, then $\boldsymbol{u} \cdot \boldsymbol{x} = 0$ on $\partial\Gamma$ as a simple consequence of the fact that $\boldsymbol{x}$ is a tangential field. As for the Neumann boundary condition in the case when $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, it is only a consequence of the formula $\partial_n \Phi = \boldsymbol{n} \cdot \boldsymbol{u}$. ∎

Similarly formulas (7.6b) and (7.6c) yield a solution of the equation $\mathbf{curl}\, \boldsymbol{u} = \boldsymbol{\psi}$:

Lemma 7.2 *Let $\boldsymbol{\psi}$ belong to $\boldsymbol{S}_T^{\lambda-1}(\Gamma)$, resp. $\boldsymbol{S}_N^{\lambda-1}(\Gamma)$.*
(i) *Then $\boldsymbol{\psi} \times \boldsymbol{x}$ belongs to $\boldsymbol{S}_N^\lambda(\Gamma)$, resp. $\boldsymbol{S}_T^\lambda(\Gamma)$.*
(ii) *We assume that $\lambda \neq -1$ and that $\operatorname{div} \boldsymbol{\psi} = 0$. If moreover $\boldsymbol{\psi}$ is homogeneous, i.e. $\boldsymbol{\psi} \in \mathring{\boldsymbol{S}}_T^{\lambda-1}(\Gamma)$, resp. $\mathring{\boldsymbol{S}}_N^{\lambda-1}(\Gamma)$, then $\boldsymbol{u}$ defined as*

$$\boldsymbol{u} = \frac{\boldsymbol{\psi} \times \boldsymbol{x}}{\lambda + 1} \quad \in \quad \mathring{\boldsymbol{S}}_N^\lambda(\Gamma) \quad \textit{resp.} \quad \mathring{\boldsymbol{S}}_T^\lambda(\Gamma), \tag{7.11}$$

solves the equation $\mathbf{curl}\, \boldsymbol{u} = \boldsymbol{\psi}$. Moreover $\operatorname{div} \boldsymbol{u} = \frac{1}{\lambda+1}\, \boldsymbol{x} \cdot \mathbf{curl}\, \boldsymbol{\psi}$.

PROOF. The regularity of $\boldsymbol{u}$ is a direct consequence of formulas (7.6b) and (7.6c). The boundary condition $\boldsymbol{n} \times (\boldsymbol{\psi} \times \boldsymbol{x}) = 0$ is satisfied if $\boldsymbol{n} \cdot \boldsymbol{\psi} = 0$ on $\partial\Gamma$ due to the equality $\boldsymbol{n} \times (\boldsymbol{\psi} \times \boldsymbol{x}) = \boldsymbol{\psi}(\boldsymbol{n} \cdot \boldsymbol{x}) - \boldsymbol{x}(\boldsymbol{n} \cdot \boldsymbol{\psi})$. And the boundary condition $\boldsymbol{n} \cdot (\boldsymbol{\psi} \times \boldsymbol{x}) = 0$ is satisfied if $\boldsymbol{n} \times \boldsymbol{\psi} = 0$ on $\partial\Gamma$ due to the equality $\boldsymbol{n} \cdot (\boldsymbol{\psi} \times \boldsymbol{x}) = \boldsymbol{x} \cdot (\boldsymbol{n} \times \boldsymbol{\psi})$. Part *(i)* is proved and part *(ii)* is now obvious. ■

The third step is the solution of the equations $\mathbf{curl}\, \boldsymbol{u} = 0$, $\operatorname{div} \boldsymbol{u} = q$, which is done with the help of formulas (7.6d)-(7.6g):

Lemma 7.3 *Let q belong to $S_1^{\lambda-1}(\Gamma)$, such that $\Delta q \in S_0^{\lambda-3}(\Gamma)$ and satisfying Dirichlet, resp. Neumann boundary conditions on $\partial\Gamma$.*
(i) *Then $2q\boldsymbol{x} + \rho^2\, \mathbf{grad}\, q$ belongs to $\boldsymbol{S}_N^\lambda(\Gamma)$, resp. $\boldsymbol{S}_T^\lambda(\Gamma)$.*
(ii) *We assume that $\lambda \neq -\frac{1}{2}$ and that $\Delta q = 0$. If moreover q is homogeneous, i.e. $q \in \mathring{S}_1^{\lambda-1}(\Gamma)$, then $\boldsymbol{u}$ defined as*

$$\boldsymbol{u} = \frac{2q\boldsymbol{x} + \rho^2\, \mathbf{grad}\, q}{4\lambda + 2} \quad \in \quad \mathring{\boldsymbol{S}}_N^\lambda(\Gamma) \quad \textit{resp.} \quad \mathring{\boldsymbol{S}}_T^\lambda(\Gamma), \tag{7.12}$$

solves the equations $\mathbf{curl}\, \boldsymbol{u} = 0$ and $\operatorname{div} \boldsymbol{u} = q$.

7.c The three types of Maxwell singularities generated by the Laplacian

In the case of plane sectors, we have seen that only two types of Maxwell singularities do exist and that they are generated by the Laplace operator: type 1, corresponding to the exponents $\frac{k\pi}{\omega} - 1$ and the singular functions of the form $\mathbf{grad}\, \Phi$ with Φ Dirichlet singularity for the Laplace operator, and type 3, corresponding to the exponents $\frac{k\pi}{\omega} + 1$. Now for three-dimensional cones, relying on the solution formulas (7.10)-(7.12) we are going to exhibit the three types in the case when they are generated by the Laplacian (Dirichlet or Neumann). In the next subsection, we will describe the remaining singularities which are generated by the *topology* of Γ.

In the following lemmas, we show the link between the sets of Maxwell singularity exponents $\Lambda_N(\Gamma)$ and $\Lambda_T(\Gamma)$ and those of the Laplacian, see $\Lambda^{\mathrm{Dir}}(\Gamma)$ and $\Lambda^{\mathrm{Neu}}(\Gamma)$ in (6.4). We also prove that the singularities of type 1, 2 and 3 can be expressed with the help of the Laplace singular functions, *cf* (6.5) for the spaces $Z_{\mathrm{Dir}}^\lambda(\Gamma)$ and $Z_{\mathrm{Neu}}^\lambda(\Gamma)$, *except in particular geometrical situations when $\lambda = -1$.*

Lemma 7.4 *We assume that $\lambda \neq -1$. Then* (i) *is equivalent to* (ii)*:*
(i) *$\boldsymbol{u} \in \boldsymbol{S}_N^\lambda(\Gamma)$ is a solution of* (7.2) *of type 1,*
(ii) *$\lambda + 1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ and $\boldsymbol{u} = \mathbf{grad}\, \Phi$ where Φ belongs to $Z_{\mathrm{Dir}}^{\lambda+1}(\Gamma)$.*
Similarly, (iii) *is equivalent to* (iv)*:*
(iii) *$\boldsymbol{u} \in \boldsymbol{S}_T^\lambda(\Gamma)$ is a solution of* (7.4) *of type 1,*
(iv) *$\lambda + 1$ belongs to $\Lambda^{\mathrm{Neu}}(\Gamma)$ and $\boldsymbol{u} = \mathbf{grad}\, \Phi$ where Φ belongs to $Z_{\mathrm{Neu}}^{\lambda+1}(\Gamma)$.*

PROOF. **1.** In a first step, we investigate the non-zero homogeneous solutions of (7.2) of type 1, i.e. solutions of

$$\mathbf{curl}\,\boldsymbol{u} = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \ \text{ in } \Gamma \qquad \text{with} \quad \boldsymbol{u} \in \mathring{\boldsymbol{S}}^{\lambda}_{N}(\Gamma).$$

Using Lemma 7.1, we immediately obtain that $\boldsymbol{u} = \mathbf{grad}\,\Phi$ with $\Phi = \frac{1}{\lambda+1}\,\boldsymbol{u}\cdot\boldsymbol{x}$. Thus

$$\Phi \in S_1^{\lambda+1} \quad \text{and} \quad \Phi = 0 \quad \text{on} \quad \partial\Gamma.$$

Moreover the condition $\operatorname{div}\boldsymbol{u} = 0$ yields that $\Delta\Phi = 0$. In other words, Φ is a Dirichlet singularity for Δ, thus $\lambda + 1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ and $\Phi \in Z^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$. The converse statement is straightforward: for any $\Phi \in Z^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$, $\boldsymbol{u}$ defined as $\mathbf{grad}\,\Phi$ is a singularity of type 1.

Concerning the magnetic boundary condition, the same arguments lead to $\mathbf{grad}\,\Phi = \boldsymbol{u}$, where Φ is still defined by (7.10) and satisfies

$$\Phi \in S_1^{\lambda+1} \quad \text{and} \quad \partial_n\Phi = 0 \quad \text{on} \quad \partial\Gamma,$$

and $\Delta\Phi = 0$. Thus $\lambda + 1$ belongs to $\Lambda^{\mathrm{Neu}}(\Gamma)$.

2. In a second step, we prove that there is no logarithmic term in any solution of type 1. It suffices to study a solution of type 1 with one logarithmic term, i.e. of the form

$$\boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{u}^1 \log\rho, \quad \text{with} \quad \boldsymbol{u}^0,\ \boldsymbol{u}^1 \in \mathring{\boldsymbol{S}}^{\lambda}_{N}(\Gamma).$$

Since $\mathbf{curl}\,\boldsymbol{u}$ is the sum of $\mathbf{curl}\,\boldsymbol{u}^1 \log\rho$ and of a field with each component in $\mathring{S}^{\lambda-1}(\Gamma)$, we deduce that $\mathbf{curl}\,\boldsymbol{u}^1 = 0$. Thus we obtain that $\boldsymbol{u}^1$ is itself a solution of type 1 of the same problem. Then instead of (7.10), we set

$$\Phi = \frac{\boldsymbol{u}\cdot\boldsymbol{x}}{\lambda+1} - \frac{\boldsymbol{u}^1\cdot\boldsymbol{x}}{(\lambda+1)^2}, \tag{7.13}$$

and we deduce from the previous remark that $\mathbf{grad}\,\Phi = \boldsymbol{u}$, and Φ satisfies the Dirichlet (or Neumann) conditions on $\partial\Gamma$ and $\Delta\Phi = 0$. Therefore Φ belongs to $Z^{\lambda+1}(\Gamma)$, but since we do not consider polynomial right hand sides here, there is no logarithmic term in Φ, hence $\boldsymbol{u}^1 = 0$. ∎

If $\boldsymbol{u}$ is a singularity of type 2, then $\boldsymbol{\psi}$ is a singularity of type 1 with a permutation of the roles of electric and magnetic boundary conditions. Moreover, when $\boldsymbol{\psi}$ is known, Lemma 7.2 provides a formula for $\boldsymbol{u}$ (note that here $\operatorname{div}\boldsymbol{\psi} = 0$, thus formula (7.11) yields a divergence free $\boldsymbol{u}$). Thus we obtain:

Lemma 7.5 *We assume that* $\lambda \notin \{-1, 0\}$. *Then* (i) *is equivalent to* (ii)*:*
(i) $\boldsymbol{u} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma)$ *is a solution of* (7.2) *of type 2,*
(ii) λ *belongs to* $\Lambda^{\mathrm{Neu}}(\Gamma)$ *and* $\mathbf{curl}\,\boldsymbol{u} = \mathbf{grad}\,\Psi$ *where* Ψ *belongs to* $Z^{\lambda}_{\mathrm{Neu}}(\Gamma)$.
Similarly (iii) *is equivalent to* (iv)*:*
(iii) $\boldsymbol{u} \in \boldsymbol{S}^{\lambda}_{T}(\Gamma)$ *is a solution of* (7.4) *of type 2,*

(iv) λ *belongs to* $\Lambda^{\mathrm{Dir}}(\Gamma)$ *and* $\mathbf{curl}\,\boldsymbol{u} = \mathbf{grad}\,\Psi$ *where* Ψ *belongs to* $Z^{\lambda}_{\mathrm{Dir}}(\Gamma)$ *.*
In each case, representatives of type 2 are given by $\boldsymbol{u} = \frac{1}{\lambda+1}\,\mathbf{grad}\,\Psi \times \boldsymbol{x}$ *.*

Finally we have directly from equations (7.2a) and (7.4a) the necessary conditions for the existence of a non-zero q and we combine lemmas 7.2 and 7.3 to obtain formulas for $\boldsymbol{\psi}$ and $\boldsymbol{u}$:

Lemma 7.6 *We assume that* $\lambda \notin \{-\frac{1}{2}, 0\}$ *. Then* (i) *is equivalent to* (ii)*:*
(i) $\boldsymbol{u} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma)$ *is a solution of* (7.2) *of type 3,*
(ii) $\lambda - 1$ *belongs to* $\Lambda^{\mathrm{Dir}}(\Gamma)$ *and* $\operatorname{div} \boldsymbol{u} = q$ *where* q *belongs to* $Z^{\lambda-1}_{\mathrm{Dir}}(\Gamma)$ *.*
Similarly (iii) *is equivalent to* (iv)*:*
(iii) $\boldsymbol{u} \in \boldsymbol{S}^{\lambda}_{T}(\Gamma)$ *is a solution of* (7.4) *of type 3,*
(iv) λ *belongs to* $\Lambda^{\mathrm{Neu}}(\Gamma)$ *and* $\operatorname{div} \boldsymbol{u} = q$ *where* q *belongs to* $Z^{\lambda-1}_{\mathrm{Neu}}(\Gamma)$ *.*
In each case, representatives of type 3 are given by $\boldsymbol{\psi} = \frac{1}{\lambda}\,\mathbf{grad}\,q \times \boldsymbol{x}$ *and by* $\boldsymbol{u} = \frac{1}{\lambda(2\lambda+1)}\Big((2\lambda-1)q\boldsymbol{x} - \rho^2\,\mathbf{grad}\,q\Big)$ *.*

Remark 7.7 For the sake of comparison, let us consider the case when Γ is a dihedron of opening ω. Then $\Lambda^{\mathrm{Dir}}(\Gamma) = \Lambda^{\mathrm{Neu}}(\Gamma) = \{\frac{k\pi}{\omega} + \ell,\ k, \ell \in \mathbb{Z},\ k \neq 0\}$, *cf* [10, Ch.18.C]. In this case, for finding explicit expressions for the singular functions, one can choose between the formulas given in Lemma 5.1 and those of Lemma 7.6. They do, however, not give the same results because of the non-trivial influence of homogeneous polynomials in the tangential variable z along the edge. ■

7.d The Maxwell singularities generated by the topology

It essentially remains to investigate the solutions of type 1 for $\lambda = -1$, i.e. the elements $\boldsymbol{u}$ in $\boldsymbol{S}^{-1}_{N}(\Gamma)$, resp. $\boldsymbol{S}^{-1}_{T}(\Gamma)$ with zero curl and divergence. The existence of such solutions depends on the topology of the spherical domain G which generates the cone Γ. We are going to prove that we have singularity spaces in $\lambda = -1$, $\boldsymbol{Z}^{-1}_{N}(\Gamma)$ and $\boldsymbol{Z}^{-1}_{T}(\Gamma)$, if and only if G is not simply connected, and that their dimensions are equal to the dimension of the homology space of G.

Lemma 7.8 *Let us assume that* G *is simply connected. If* $\boldsymbol{u}$ *belongs to* $\boldsymbol{S}^{-1}_{N}(\Gamma)$ *, resp.* $\boldsymbol{S}^{-1}_{T}(\Gamma)$ *and satisfies* $\mathbf{curl}\,\boldsymbol{u} = 0$ *and* $\operatorname{div} \boldsymbol{u} = 0$ *, then* $\boldsymbol{u} = 0$ *.*

PROOF. Since Γ is simply connected, we derive from the condition $\mathbf{curl}\,\boldsymbol{u} = 0$ that $\boldsymbol{u}$ is the gradient of some function Φ. Then we use a formula of integration of $\boldsymbol{u}$ along paths like (5.8): we fix $\boldsymbol{x}_0 \in \Gamma$ and write in polar coordinates $(\rho, \vartheta) = (|\boldsymbol{x}|, \boldsymbol{x}/|\boldsymbol{x}|)$ and $(\rho_0, \vartheta_0) = (|\boldsymbol{x}_0|, \boldsymbol{x}_0/|\boldsymbol{x}_0|)$:

$$\Phi(\rho, \vartheta) - \Phi(\rho_0, \vartheta_0) = \int_{\rho_0}^{\rho} \frac{\boldsymbol{x}_0}{\rho_0} \cdot \boldsymbol{u}(\rho', \vartheta_0)\, d\rho' \ + \ \rho \int_{\gamma(\vartheta_0, \vartheta)} \boldsymbol{u}(\rho, \vartheta') \cdot d\vartheta', \tag{7.14}$$

where the second integral is a path integral along a curve $\gamma(\vartheta_0,\vartheta)$ from ϑ_0 to ϑ in G. From (7.14), we find that Φ belongs to $S^0_1(\Gamma)$. The condition $\operatorname{div}\boldsymbol{u}=0$ yields that $\Delta\Phi=0$ and the boundary conditions on $\boldsymbol{u}$ give either the Dirichlet conditions on Φ, or the Neumann condition. In the first case, we find that $\Phi=0$ since the eigenvalues μ_j^{Dir} are all >0, and in the second case, we find that Φ is a constant, thus in any case $\boldsymbol{u}=0$. ■

If the spherical domain G is not simply connected, its boundary ∂G is not connected. Let $\partial_j G$, $j=1,\ldots,J+1$ be its connected components ($J\geq 1$). We assume that G itself is connected (if not, the cones corresponding to each of its connected components can be considered separately). Then there exist J regular and non-intersecting cuts σ_j, $j=1,\ldots,J$ such that $G^0:=G\setminus\cup_{j=1}^J\sigma_j$ is simply connected.

The singularities of degree -1 that we are investigating are closely linked with the kernels (7.15) and (7.16) of the tangential curl and divergence, $\operatorname{curl}_\top$ and $\operatorname{div}_\top$. These operators are tangential to the sphere $\mathbb{S}^2$ and can be defined with the help of the usual **curl** and div on three-dimensional fields: we first introduce $L^2_\top(G)$ as the subspace of $L^2(G)^3$ spanned by the fields $\boldsymbol{v}$ tangential to the sphere, i.e. satisfying $\boldsymbol{v}\cdot\boldsymbol{x}=0$. If $\boldsymbol{v}$ belongs to $L^2_\top(G)$, we can introduce $\widehat{\boldsymbol{v}}$ as any homogeneous extension of $\boldsymbol{v}$ to the cone Γ: we fix any μ and $\widehat{\boldsymbol{v}}(\rho,\vartheta)$ is defined as $\rho^\mu\boldsymbol{v}(\vartheta)$. Then

- $\operatorname{div}_\top\boldsymbol{v}$ is the restriction on $\rho=1$ of $\operatorname{div}\widehat{\boldsymbol{v}}$,
- $\operatorname{curl}_\top\boldsymbol{v}$ is the restriction on $\rho=1$ of $\boldsymbol{x}\cdot\mathbf{curl}\,\widehat{\boldsymbol{v}}$.

Then the two kernels are defined in a classical way:

$$K_N(G)=\Big\{\boldsymbol{v}\in L^2_\top(G)\mid \operatorname{curl}_\top\boldsymbol{v}=0,\ \operatorname{div}_\top\boldsymbol{v}=0\ \text{ in } G,\ \boldsymbol{v}\times\boldsymbol{n}=0\ \text{ on } \partial G\Big\}\tag{7.15}$$

and

$$K_T(G)=\Big\{\boldsymbol{v}\in L^2_\top(G)\mid \operatorname{curl}_\top\boldsymbol{v}=0,\ \operatorname{div}_\top\boldsymbol{v}=0\ \text{ in } G,\ \boldsymbol{v}\cdot\boldsymbol{n}=0\ \text{ on } \partial G\Big\}.\tag{7.16}$$

Their description involves the tangential gradient $\mathbf{grad}_\top$, and also (alternatively) the tangential vectorial curl $\mathbf{curl}_\top$, which are defined for any scalar function Φ in $L^2(G)$ with the help of any homogeneous extension $\widehat{\Phi}$ of Φ to Γ as follows:

- $\mathbf{grad}_\top\Phi$ is the restriction on $\rho=1$ of $\mathbf{grad}\,\widehat{\Phi}-(\mathbf{grad}\,\widehat{\Phi}\cdot\boldsymbol{x})\,\boldsymbol{x}$,
- $\mathbf{curl}_\top\Phi$ is the restriction on $\rho=1$ of $\mathbf{curl}(\widehat{\Phi}\,\boldsymbol{x})$.

There holds the following description of the spaces $K_N(G)$ and $K_T(G)$ (see [6] for a classical presentation and [1] for the case of less regular domains). In the definitions (7.17) and (7.18) below, the c_j denote arbitrary constant functions, n_j is a unitary normal to σ_j in $\mathbb{S}^2$ and $[\,\cdot\,]_{\sigma_j}$ the jump across σ_j along n_j:

Lemma 7.9 (i) *The space $K_N(G)$ is generated by the tangential gradients $\mathbf{grad}_\top\Phi$ where Φ spans the space $P_{\mathrm{Dir}}(G)$ defined as*

$$P_{\mathrm{Dir}}(G)=\Big\{\Phi\in H^1(G)\mid\ \begin{array}{l}\Delta_G\Phi=0\ \ in\ G,\\ \Phi=c_j\ \ on\ \partial_j G,\ 1\leq j\leq J+1\Big\}.\end{array}\tag{7.17}$$

The dimension of $P_{\mathrm{Dir}}(G)$ *is* $J+1$ *and the dimension of* $K_N(G)$ *is* J *.*

(ii) *The space* $K_T(G)$ *is generated by the* L^2 *extensions* $\widetilde{\mathbf{grad}}_{\top}\Phi$ *to* G *of the tangential gradients* $\mathbf{grad}_{\top}\Phi$ *on* G^0 *where* Φ *spans the space* $P_{\mathrm{Neu}}(G)$ *defined as*

$$P_{\mathrm{Neu}}(G) = \Big\{\Phi \in H^1(G^0) \mid \quad \Delta_G\Phi = 0 \quad \textit{in } G,\ \partial_n\Phi = 0 \quad \textit{on } \partial G, \\ [\Phi]_{\sigma_j} = c_j,\ [\partial_{n_j}\Phi]_{\sigma_j} = 0,\ 1 \le j \le J\Big\}. \tag{7.18}$$

The dimension of $P_{\mathrm{Neu}}(G)$ *is* J *and the dimension of* $K_T(G)$ *is* J *, too.*

Relying on the definitions of the "tangential" operators $\mathbf{curl}_{\top}$, $\mathbf{grad}_{\top}$, $\operatorname{curl}_{\top}$, $\operatorname{div}_{\top}$ and on the relations (7.6b)-(7.6d), it is easy to show that for any $\Phi \in L^2(G)$ and any $\boldsymbol{v} \in L^2_{\top}(G)$

$$\mathbf{curl}_{\top}\Phi = (\mathbf{grad}_{\top}\Phi) \times \boldsymbol{x}, \quad \operatorname{div}_{\top}\boldsymbol{v} = -\operatorname{curl}_{\top}(\boldsymbol{v} \times \boldsymbol{x}), \quad \operatorname{curl}_{\top}\boldsymbol{v} = \operatorname{div}_{\top}(\boldsymbol{v} \times \boldsymbol{x}).$$

Then, we can prove the following

Corollary 7.10 *The space* $K_N(G)$ *is generated by the extended tangential curls* $\widetilde{\mathbf{curl}}_{\top}\Phi$ *where* Φ *spans the space* $P_{\mathrm{Neu}}(G)$ *and the space* $K_T(G)$ *is generated by* $\mathbf{curl}_{\top}\Phi$ *where* Φ *spans the space* $P_{\mathrm{Dir}}(G)$ *. Moreover, we have the relations*

$$K_T = \boldsymbol{x} \times K_N \quad \textit{and} \quad K_N = \boldsymbol{x} \times K_T.$$

Thus, the kernels $K_N(G)$ and $K_T(G)$ are gradients (or curls) of harmonic functions belonging to the spaces $P_{\mathrm{Dir}}(G)$ and $P_{\mathrm{Neu}}(G)$. We extend these spaces to homogeneous functions of degree 0 on the cone Γ and thus define the spaces $P_{\mathrm{Dir}}(\Gamma)$ and $P_{\mathrm{Neu}}(\Gamma)$. We note that any $\Phi \in P_{\mathrm{Dir}}(\Gamma)$ has its traces constant on each connected component of $\partial\Gamma$ and similarly that the jumps of any $\Phi \in P_{\mathrm{Neu}}(\Gamma)$ across the cuts Σ_j of Γ corresponding to σ_j are constant too.

For any $\Phi \in P_{\mathrm{Dir}}(\Gamma)$, the gradient $\mathbf{grad}\,\Phi$ is an homogeneous function of degree -1 whose radial component is 0: we have

$$\mathbf{grad}\,\Phi(\rho,\vartheta) = \frac{1}{\rho}\ \mathbf{grad}_{\top}\Phi(\vartheta).$$

The field $\boldsymbol{u} = \mathbf{grad}\,\Phi$ belongs to $\boldsymbol{S}_N^{-1}(\Gamma)$ and satisfies $\mathbf{curl}\,\boldsymbol{u} = 0$ like all gradients, and $\operatorname{div}\boldsymbol{u} = \Delta\Phi = \rho^{-2}\Delta_G\Phi = 0$ by construction. Similarly for $\Phi \in P_{\mathrm{Neu}}(\Gamma)$, the extended gradient $\boldsymbol{u} = \widetilde{\mathbf{grad}}\Phi$ belongs to $\boldsymbol{S}_T^{-1}(\Gamma)$ and satisfies $\mathbf{curl}\,\boldsymbol{u} = 0$ and $\operatorname{div}\boldsymbol{u} = 0$.

Lemma 7.11 *Let us assume that* G *is not simply connected. Then* $\boldsymbol{Z}_N^{-1}(\Gamma)$ *is the space of the fields of the form* $\boldsymbol{u} = \mathbf{grad}\,\Phi$ *, where* $\Phi \in P_{\mathrm{Dir}}(\Gamma)$ *. Correspondingly,* $\boldsymbol{Z}_T^{-1}(\Gamma)$ *is the space of the fields of the form* $\boldsymbol{u} = \widetilde{\mathbf{grad}}\Phi$ *, where* $\Phi \in P_{\mathrm{Neu}}(\Gamma)$ *.*

Proof. We have just proved that any field of the form $\boldsymbol{u} = \mathbf{grad}\,\Phi$, where Φ is a non-zero element of $P_{\mathrm{Dir}}(\Gamma)$ is a non-trivial element of $\boldsymbol{Z}_N^{-1}(\Gamma)$. Conversely let $\boldsymbol{u}$ belong to $\boldsymbol{Z}_N^{-1}(\Gamma)$. Then, for a non-zero field $\boldsymbol{u}_P$, $P \geq 0$, we have

$$\boldsymbol{u} = \rho^{-1}\Big(\boldsymbol{u}_0 + \cdots + \log^P\rho\ \boldsymbol{u}_P\Big).$$

Since $\mathbf{curl}\,\boldsymbol{u} = 0$, $\boldsymbol{u}$ is a gradient $\mathbf{grad}\,\Phi$ in the simply connected domain Γ^0 generated by the spherical domain G^0. Using formula (7.14) with paths $\gamma(\vartheta_0, \vartheta)$ contained in G^0, we obtain that Φ belongs to $S_1^0(\Gamma^0)$ and can be expanded into

$$\Phi = \Phi_0 + \cdots + \log^Q\rho\ \Phi_Q, \quad \text{with} \quad P \leq Q \leq P+1.$$

Since $\boldsymbol{u}\times\boldsymbol{n}$ is zero on $\partial\Gamma$, the traces of Φ on $\partial\Gamma$ are constant on each of its connected components $\partial_j\Gamma$. Thus the traces of Φ_0 are constant on each $\partial_j G$ and the traces of Φ_q for $q \geq 1$ are zero. Since $\mathbf{curl}\,\boldsymbol{u}$ and $\operatorname{div}\boldsymbol{u}$ are zero in Γ, the jumps of $\boldsymbol{u}$ across the cuts Σ_j generated by σ_j are zero. Thus $[\Phi]_{\Sigma_j}$ is constant and $[\partial_{n_j}\Phi]_{\Sigma_j}$ is zero. With the conditions on the traces on $\partial_j\Gamma$, this yields that $[\Phi]_{\Sigma_j}$ is zero. Moreover $\operatorname{div}\boldsymbol{u} = 0$ in Γ gives $\Delta_G\Phi_Q = 0$.

Therefore Φ_Q belongs to $P_{\mathrm{Dir}}(G)$, and if $Q \geq 1$ we have moreover that the traces of Φ_Q on ∂G are all zero, thus $\Phi_Q = 0$. Whence $Q = 0$ and Φ_0 belongs to $P_{\mathrm{Dir}}(G)$, so $\boldsymbol{u}$ has the desired form.

The proof for $\boldsymbol{Z}_T^{-1}(\Gamma)$ is similar. ■

7.e Corner singularities: a synthesis

Let us consider the electric boundary condition. According to the general theory stated in §4.c, we have to consider the subset of $\Lambda_N(\Gamma)$ satisfying (with χ a cut-off function which is equal to 1 in a neighborhood of the corner of Γ):

$$-\tfrac{3}{2} < \lambda \quad \text{and} \quad \exists \boldsymbol{u} \in \boldsymbol{Z}_N^{\lambda}(\Gamma),\ \boldsymbol{u} \neq 0, \quad \chi\boldsymbol{u} \in X_N(\Gamma),\ \operatorname{div}(\chi\boldsymbol{u}) \in H^1(\Gamma). \tag{7.19}$$

We examine the effect of condition (7.19) on each of the singularities of types 1, 2 and 3, *cf* Lemmas 7.4 - 7.6 and 7.11.

Type 1 In this case $\lambda + 1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ (subtype 1_Δ) or $\lambda = -1$ if G is not simply connected (subtype 1_{Top}): according to (6.4), as $\mu_1^{\mathrm{Dir}} > 0$, $\Lambda^{\mathrm{Dir}}(\Gamma) \cap [-1, 0]$ is empty. Thus λ avoids $[-2, -1]$ for type 1_Δ. Whence $\lambda > -1$. The corresponding singularities are $\mathbf{grad}\,\Phi$ with Φ either in $Z_{\mathrm{Dir}}^{\lambda+1}(\Gamma)$ for type 1_Δ or in $P_{\mathrm{Dir}}(\Gamma)$ for type 1_{Top}.

Type 2 In this case $\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{u}$ is a singularity of type 1 with magnetic boundary conditions and condition (7.19) on $\boldsymbol{u}$ requires that $\chi\boldsymbol{\psi}$ belongs to $L^2(\Gamma)$, thus $\lambda - 1 > -\frac{3}{2}$, i.e. $\lambda > -\frac{1}{2}$. Then we have the subtype 2_Δ corresponding to $\boldsymbol{\psi} = \mathbf{grad}\,\Psi$ with $\Psi \in Z_{\mathrm{Neu}}^{\lambda}(\Gamma)$ and $\lambda \in \Lambda^{\mathrm{Neu}}(\Gamma)$ (thus $\lambda > 0$) and the subtype 2_{Top} corresponding to $\boldsymbol{\psi} = \mathbf{grad}\Psi$ with $\Psi \in P_{\mathrm{Neu}}(\Gamma)$ and $\lambda = 0$ (if G is not simply connected).

Type 3 In this case $q = \operatorname{div} \boldsymbol{u}$ is a Dirichlet singularity of the Laplace operator and $\lambda - 1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$. Thus condition (7.19) on $\boldsymbol{u}$ requires that χq belongs to $H^1(\Gamma)$, thus $\lambda > \frac{1}{2}$, whence $\lambda > 1$.

Thus the most singular among the admissible elements of the $\boldsymbol{Z}_N^\lambda(\Gamma)$ is either

$$\boldsymbol{u} = \mathbf{grad}\,\Phi \quad \text{with} \quad \Phi \in Z_{\mathrm{Dir}}^{\lambda+1}(\Gamma), \quad \lambda = -\tfrac{3}{2} + \sqrt{\mu_1^{\mathrm{Dir}} + \tfrac{1}{4}}.$$

if Γ is simply connected, or the singularity of type 1_{Top} if not.

We summarize all the results in the following table, where we omit the reference to the cone Γ in the notation of spaces:

Type	λ	$>$	Generator	$\boldsymbol{u}$	$\boldsymbol{\psi}$	q
1_Δ	$\lambda + 1 \in \Lambda^{\mathrm{Dir}}$	-1	$\Phi \in Z_{\mathrm{Dir}}^{\lambda+1}$	$\mathbf{grad}\,\Phi$	0	0
2_Δ	$\lambda \in \Lambda^{\mathrm{Neu}}$	0	$\Psi \in Z_{\mathrm{Neu}}^{\lambda}$	$\dfrac{\mathbf{grad}\,\Psi \times \boldsymbol{x}}{\lambda + 1}$	$\mathbf{grad}\,\Psi$	0
3_Δ	$\lambda - 1 \in \Lambda^{\mathrm{Dir}}$	1	$q \in Z_{\mathrm{Dir}}^{\lambda-1}$	$\dfrac{(2\lambda - 1)q\boldsymbol{x} - \rho^2\,\mathbf{grad}\,q}{\lambda(2\lambda+1)}$	$\dfrac{\mathbf{grad}\,q \times \boldsymbol{x}}{\lambda}$	q
1_{Top}	-1		$\Phi \in P_{\mathrm{Dir}}$	$\mathbf{grad}\,\Phi$	0	0
2_{Top}	0		$\Psi \in P_{\mathrm{Neu}}$	$\widetilde{\mathbf{grad}}\Psi \times \boldsymbol{x}$	$\widetilde{\mathbf{grad}}\Psi$	0

Alternative formulation

2_{Top}	0		$\Phi \in P_{\mathrm{Dir}}$	$\rho\,\mathbf{grad}\,\Phi$	$\dfrac{\boldsymbol{x}}{\rho} \times \mathbf{grad}\,\Phi$	0

Table 1

This alternative formulation is obtained with the help of Corollary 7.10. The adaptation of this table to magnetic boundary conditions is left to the reader.

Going back to the primitive Maxwell equations (1.1), we see that for a regular current density $\boldsymbol{J}$, the divergences of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are regular too, thus only the singularities of types 1 and 2 can occur and they exchange each other between the electric and magnetic fields (here λ denotes the degree of homogeneity of the generator and is either the degree of $\boldsymbol{E}$ or $\boldsymbol{H}$):

Type	Generator	λ	$\boldsymbol{E}$	$\boldsymbol{H}$
Δ (electric)	$\Phi \in Z^{\lambda}_{\mathrm{Dir}}$	$\lambda \in \Lambda^{\mathrm{Dir}}$	$\mathbf{grad}\,\Phi$	$\dfrac{\mathbf{grad}\,\Phi \times \boldsymbol{x}}{i\omega(\lambda+1)}$
Δ (magnetic)	$\Psi \in Z^{\lambda}_{\mathrm{Neu}}$	$\lambda \in \Lambda^{\mathrm{Neu}}$	$-\dfrac{\mathbf{grad}\,\Psi \times \boldsymbol{x}}{i\omega(\lambda+1)}$	$\mathbf{grad}\,\Psi$
Top (electric)	$\Phi \in P_{\mathrm{Dir}}$	0	$\mathbf{grad}\,\Phi$	$\dfrac{\mathbf{grad}\,\Phi \times \boldsymbol{x}}{i\omega}$
Top (magnetic)	$\Psi \in P_{\mathrm{Neu}}$	0	$-\dfrac{\widetilde{\mathbf{grad}}\Psi \times \boldsymbol{x}}{i\omega}$	$\widetilde{\mathbf{grad}}\Psi$

Table 2

This table gives the *principal parts* of the singularities (as can be seen from (1.2) or (1.7) the operators are not homogeneous and according to the general theory [15, 10] the singularities themselves have an asymptotic expansion).

8 Conclusions

Taking advantage of the information about admissible exponents and singularities collected in sections 5 and 7, we are now able to give more specific versions of the regularity and singularity theorems 4.1 and 4.2.

For any edge $\boldsymbol{e} \in \mathscr{E}$ and $\sigma \geq -1$ (below, k denotes any integer ≥ 1, with the requirement that k is odd if $\omega_e = 2\pi$), the set of admissible exponents is:

$$\Lambda^{\sigma}_{N}(\Gamma_e) = \left\{ \tfrac{k\pi}{\omega_e} - 1 \leq \sigma \mid \ k \geq 2 \text{ if } \omega_e = \tfrac{\pi}{2} \right\} \bigcup \left\{ \tfrac{k\pi}{\omega_e} \leq \sigma \right\} \bigcup \left\{ \tfrac{k\pi}{\omega_e} + 1 \leq \sigma \right\}. \qquad (8.1)$$

For any corner $\boldsymbol{c} \in \mathscr{C}$ and $\sigma \geq -1$, $\lambda_j^{\rm Dir}(\Gamma_c)$ and $\lambda_j^{\rm Neu}(\Gamma_c)$ denote the Dirichlet and Neumann Laplace singular exponents on Γ_c, *cf* (6.4)

$$\lambda_j^{\rm Dir}(\Gamma_c) = -\tfrac{1}{2} + \sqrt{\mu_j^{\rm Dir} + \tfrac{1}{4}} \quad \text{and} \quad \lambda_j^{\rm Neu}(\Gamma_c) = -\tfrac{1}{2} + \sqrt{\mu_j^{\rm Neu} + \tfrac{1}{4}},$$

and the set of admissible exponents is

$$\begin{aligned}\Lambda_N^\sigma(\Gamma_c) = \Big\{ \lambda \pm 1 \leq \sigma - \tfrac{1}{2} \mid \lambda = \lambda_j^{\rm Dir}(\Gamma_c) \Big\} \cup \Big\{ \lambda \leq \sigma - \tfrac{1}{2} \mid \lambda = \lambda_j^{\rm Neu}(\Gamma_c) \Big\} \\ \cup \Big\{ -1, 0 \leq \sigma - \tfrac{1}{2} \mid \text{ if } G_c \text{ is not simply connected} \Big\}.\end{aligned} \tag{8.2}$$

Let now be

$$\sigma^E = \min_{e \in \mathscr{E}} \left\{ \begin{array}{ll} \frac{\pi}{\omega_e} & \text{if } \omega_e \neq \frac{\pi}{2} \\ 3 & \text{if } \omega_e = \frac{\pi}{2} \end{array} \right\} \tag{8.3}$$

and

$$\sigma^{\rm Dir} = \min_{c \in \mathscr{C}} \lambda_1^{\rm Dir}(\Gamma_c) \quad \text{and} \quad \sigma^{\rm Neu} = \min_{c \in \mathscr{C}} \lambda_1^{\rm Neu}(\Gamma_c). \tag{8.4}$$

8.a Regularity

Let $s \in [1, \frac{5}{2})$ and $\tau \in (0, s+1]$ (τ plays the role of $1 + \sigma$). For any data $\boldsymbol{f} \in H^{s-1}(\Omega)^3$ let $\boldsymbol{u}$ be the solution of problem (4.1). As a corollary of Theorem 4.1, $\boldsymbol{u} \in H^\tau(\Omega)^3$ for all τ, $\tau < \min\{\sigma^E,\ \sigma^{\rm Dir} + \frac{1}{2},\ \sigma^{\rm Neu} + \frac{3}{2}\}$. More precisely, according to the geometrical properties of Ω we have:

(i) If Ω has screen parts, or is not locally simply connected, then $\boldsymbol{u} \in H^\tau(\Omega)^3$ for all $\tau < \frac{1}{2}$.

(ii) If Ω has no screen parts and is locally simply connected, but is not convex, then $\boldsymbol{u} \in H^\tau(\Omega)^3$ for all τ, $\frac{1}{2} < \tau < \min\{\sigma^E,\ \sigma^{\rm Dir} + \frac{1}{2}\} < 1$.

(iii) If Ω is convex, the monotonicity of Dirichlet eigenvalues allows then to prove that $\sigma^E \leq \sigma^{\rm Dir}$, thus $\boldsymbol{u} \in H^\tau(\Omega)^3$ for all τ, $1 < \tau < \min\{\sigma^E,\ \sigma^{\rm Neu} + \frac{3}{2}\}$.

(iv) If Ω is a parallelepiped, $\boldsymbol{u} \in H^\tau(\Omega)^3$ for all $\tau < 3$.

8.b Singularities

In this whole subsection $s \in [1, \frac{5}{2})$ and $\sigma \in [0, s]$, the data $\boldsymbol{f}$ belongs to $H^{s-1}(\Omega)^3$ and $\boldsymbol{u}$ is the solution of problem (4.1). We are going to give three forms (more and more particular) of the expansion (4.9).

A. If $\sigma < \sigma^E$ and $\sigma - \frac{1}{2} \notin \Lambda_N^\sigma(\Gamma_c)$ for any $\boldsymbol{c} \in \mathscr{C}$, the splitting (4.9) can be written as

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_{{\rm reg},\sigma} + \sum_{c \in \mathscr{C}} \sum_{\lambda \in \Lambda_N^\sigma(\Gamma_c)} \gamma_c^\lambda\, \chi_c(\rho_c)\, \boldsymbol{U}_c^\lambda(\rho_c, \vartheta_c) \\ + \sum_{e \in \mathscr{E}} \sum_{k \in K_e^\sigma} \mathscr{K}[\gamma_{e,k}]\, \chi_e(\rho_e)\, \mathbf{grad}_e \Big(\rho_e^{k\pi/\omega_e} \sin \tfrac{k\pi}{\omega_e} \theta_e \Big)\end{aligned} \tag{8.5}$$

with $\boldsymbol{u}_{\mathrm{reg},\sigma} \in H^{\sigma+1}(\Omega)^3$ and the edge coefficients $\gamma_{e,k} \in \mathbb{V}_{-\sigma}^{s+1-k\pi/\omega_e}(\boldsymbol{e})$.

Here, for any $\boldsymbol{c} \in \mathscr{C}$, with $\Phi_j^{\mathrm{Dir}} = \rho^{\lambda_j^{\mathrm{Dir}}} \varphi_j^{\mathrm{Dir}}(\vartheta)$ and a similar notation for Φ_j^{Neu}:

if $\lambda = \lambda_j^{\mathrm{Dir}} - 1$, $\quad \boldsymbol{U}_c^\lambda = \mathbf{grad}\, \Phi_j^{\mathrm{Dir}}$,
if $\lambda = \lambda_j^{\mathrm{Neu}}$, $\quad \boldsymbol{U}_c^\lambda = \mathbf{grad}\, \Phi_j^{\mathrm{Neu}} \times \boldsymbol{x}$,
if $\lambda = \lambda_j^{\mathrm{Dir}} + 1$, $\quad \boldsymbol{U}_c^\lambda = (2\lambda - 1)\Phi_j^{\mathrm{Dir}}\boldsymbol{x} - \rho^2\, \mathbf{grad}\, \Phi_j^{\mathrm{Dir}}$,
if $\lambda = -1$ or 0, $\quad \boldsymbol{U}_c^\lambda$ is the corresponding topological singularity (case when G_c is not simply connected).

For any $\boldsymbol{e} \in \mathscr{E}$,

$$K_e^\sigma = \Big\{ k \in \{1,2\} \mid \tfrac{k\pi}{\omega_e} < \sigma + 1, \text{ and } k \neq 1 \text{ if } \omega_e = \tfrac{\pi}{2},\ k \neq 2 \text{ if } \omega_e = 2\pi \Big\}$$

and $\mathbf{grad}_e$ is the gradient associated with the variables $\tilde{y}_e = y_e/d_e$ and z_e.*

B. We assume now that Ω is locally simply connected and that $\sigma < \sigma^E$ as before, and moreover that $\sigma < \min\{\sigma^{\mathrm{Neu}} + \frac{1}{2},\ \sigma^{\mathrm{Dir}} + \frac{3}{2}\}$. Then the above splitting (8.5) only involves gradients in the singular part and can be written as

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_{\mathrm{reg},\sigma} + \sum_{c\in\mathscr{C}} \sum_{j\in J_c^\sigma} \gamma_{c,j}\, \chi_c(\rho_c)\ \mathbf{grad}\, \Phi_{c,j}^{\mathrm{Dir}}(\rho_c, \vartheta_c) \\ + \sum_{e\in\mathscr{E}} \sum_{k\in K_e^\sigma} \mathscr{K}[\gamma_{e,k}]\, \chi_e(\rho_e)\ \mathbf{grad}_e\, \Phi_{e,k}^{\mathrm{Dir}}(\rho_e, \theta_e)\end{aligned} \tag{8.6}$$

with $J_c^\sigma = \{j \geq 1,\ \lambda_{c,j}^{\mathrm{Dir}} < \sigma + \frac{1}{2}\}$ and the obvious definition for $\Phi_{e,k}^{\mathrm{Dir}}$.

C. If Ω is locally simply connected and if we take $\sigma = 0$, $\boldsymbol{u}_{\mathrm{reg},0}$ is the H^1 regular part of $\boldsymbol{u}$ (which was also called $\boldsymbol{u}_{(H)}$ in the section 3 devoted to polygons) and denoting $\mathscr{C}_0 = \{c \in \mathscr{C},\ \lambda_{c,1}^{\mathrm{Dir}} < \frac{1}{2}\}$ and $\mathscr{E}_0 = \{e \in \mathscr{E},\ \omega_e > \pi\}$, the splitting (8.6) takes the simplified form

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_{\mathrm{reg},0} + \sum_{c\in\mathscr{C}_0} \gamma_c\, \chi_c(\rho_c)\ \mathbf{grad}\, \Phi_{c,1}^{\mathrm{Dir}}(\rho_c, \vartheta_c) \\ + \sum_{e\in\mathscr{E}_0} \mathscr{K}[\gamma_e]\, \chi_e(\rho_e)\ \mathbf{grad}_e\, \Phi_{e,1}^{\mathrm{Dir}}(\rho_e, \theta_e).\end{aligned} \tag{8.7}$$

Remark 8.1 In the splittings (8.6) and (8.7), the singular generators can also be expressed as curls since for any harmonic and homogeneous function Φ of degree λ, *cf* (7.6b):

$$(\lambda + 1)\, \mathbf{grad}\, \Phi = \mathbf{curl}(\mathbf{grad}\, \Phi \times \boldsymbol{x})$$

and

$$\mathbf{grad}_e(\rho_e^\lambda \sin \lambda\theta_e) = \mathbf{curl}_e(\rho_e^\lambda \cos \lambda\theta_e).$$

with $\mathbf{curl}_e$ the two-dimensional vectorial curl in the $\tilde{y}_e$ plane, completed by a zero tangential component along the edge. ∎

*We recall that y_e are cartesian coordinates in the normal plane to $\boldsymbol{e}$, z_e is a coordinate tangent to $\boldsymbol{e}$ and d_e is equivalent to the distance to the ends of $\boldsymbol{e}$.

Another interesting question in the framework of the splittings (8.6) and (8.7), is to know whether it is possible to write the singular parts as a gradients in a global way.

Lemma 8.2 *Let* $\sigma \in [0, \sigma^E)$ *and* $\boldsymbol{c} \in \mathscr{C}$ *. Let* $\Phi = \Phi^{\mathrm{Dir}}_{c,j}$ *be a singularity of the Laplace Dirichlet problem on* Γ_c *. Then*

$$\chi_c \,\mathbf{grad}\,\Phi - \mathbf{grad}(\chi_c \Phi) \in H^{\sigma+1}(\Omega) \tag{8.8}$$

and $\Psi := \chi_c \Phi$ *belongs to* $\mathring{H}^1(\Omega)$ *and is such that* $\Delta\Psi$ *is in* $H^\sigma(\Omega)$ *.*

Remark 8.3 The limitation of the regularity in (8.8) and for $\Delta\Psi$ comes only from the brutal cut-off of the edge asymptotics of Φ away from the corner. A refined cut-off procedure [10, §16.C], would yield a similar statement without any limitation on σ . ■

Lemma 8.4 *Let* $\sigma \in [0, \sigma^E)$ *and* $\boldsymbol{e} \in \mathscr{E}$ *. Let* $\Phi = \Phi^{\mathrm{Dir}}_{e,k}$ *be a singularity of the Laplace Dirichlet problem on* Γ_e *, with* $\frac{k\pi}{\omega_e} - 1 < \sigma$ *. Let for* $s \geq \sigma$ *,* $\gamma \in \mathbb{V}^{s+1-k\pi/\omega_e}_{-\sigma}(\boldsymbol{e})$ *. Then, with the coordinates* $\tilde{y}_e = y_e/d_e$

$$\mathscr{K}[\gamma]\,\chi_e(\tilde{y}_e)\,\mathbf{grad}_e\,\Phi(\tilde{y}_e) - \mathbf{grad}\Big(\mathscr{K}[d_e\,\gamma]\,\chi_e(\tilde{y}_e)\,\Phi(\tilde{y}_e)\Big) \in H^{\sigma+1}(\Omega) \tag{8.9}$$

and $\Psi := \mathscr{K}[d_e\,\gamma]\,\chi_e\Phi$ *belongs to* $\mathring{H}^1(\Omega)$ *and is such that* $\Delta\Psi$ *is in* $H^\sigma(\Omega)$ *.*

Remark 8.5 Beyond what could be done by the introduction of correct "shadow" terms, it is impossible to avoid the limitation of the regularity by the weight $-\sigma$ in the space containing the edge coefficient γ . This implies that, if we apply such a refined statement to the edge terms in (8.5), we have a sharp limitation by the smallest corner exponents which does not correspond to gradients ($\sigma^{\mathrm{Neu}} + \frac{1}{2}$ and $\sigma^{\mathrm{Dir}} + \frac{3}{2}$). ■

As a consequence of the expansion (8.6) and of the two previous lemmas, we obtain

Theorem 8.6 *Let* Ω *be locally simply connected,* $\sigma < \min\{\sigma^E,\ \sigma^{\mathrm{Neu}} + \frac{1}{2},\ \sigma^{\mathrm{Dir}} + \frac{3}{2}\}$ *and* $s \geq \sigma$ *. Then for any data* $\boldsymbol{f} \in H^{s-1}(\Omega)^3$ *the solution* $\boldsymbol{u}$ *of problem* (4.1) *can be split in the following way*

$$\boldsymbol{u} = \boldsymbol{u}^*_{\mathrm{reg},\sigma} + \mathbf{grad}\,\Psi \tag{8.10}$$

where $\boldsymbol{u}^*_{\mathrm{reg},\sigma} \in H^{\sigma+1}(\Omega)^3$ *and* Ψ *can be written as*

$$\begin{aligned}\Psi = \sum_{c\in\mathscr{C}} \sum_{j\in J^\sigma_c} \gamma_{c,j}\,\chi_c(\rho_c)\,\Phi^{\mathrm{Dir}}_{c,j}(\rho_c,\vartheta_c) \\ + \sum_{e\in\mathscr{E}} \sum_{k\in K^\sigma_e} \mathscr{K}[d_e\,\gamma_{e,k}]\,\chi_e(\rho_e)\,\Phi^{\mathrm{Dir}}_{e,k}(\rho_e,\theta_e).\end{aligned} \tag{8.11}$$

Here $\Psi \in \mathring{H}^1(\Omega)$ *satisfies* $\Delta\Psi \in H^\sigma(\Omega)$ *.*

When applied with $\sigma = 0$, the above statement can be compared with the splitting of any element of X_N in the sum of an element of H_N and of a term $\mathbf{grad}\,\Phi$ with $\Phi \in \mathring{H}^1(\Omega)$ such that $\Delta\Phi \in L^2(\Omega)$, *cf* [3, 4, 11].

8.c Singularities in H^1

We conclude by some remarks on the regularity of the variational problem (2.8) posed in $H_N \subset H^1(\Omega)^3$. Very similar remarks hold for the saddle-point variant (2.4), posed in $H_N \times L^2(\Omega)$. In section 3.f, this question was discussed for polygons. We saw that the set of exponents $\Lambda^s_{(H)}(\Gamma)$ was, in general, different from $\Lambda^s_{(X)}(\Gamma)$. Still these exponents were related to those of the Dirichlet problem for the Laplace operator in a simple way.

Consider now the case of a three-dimensional (pseudo-)polyhedral corner $\boldsymbol{c}$. If the cone $\Gamma_{\boldsymbol{c}}$ is convex, then the X_N-singular functions belong to H^1, and for the H_N-singular functions, the divergence has H^1 regularity. Thus the two problems have the same singularities near the corner $\boldsymbol{c}$.

We suppose therefore that the cone $\Gamma_{\boldsymbol{c}}$ is not convex. There exist general results on the exponents of the singular functions for this problem in the case of a Lipschitz cone (Kozlov - Mazya - Rossmann [16]). For instance, there is a strip $-1 \leq \operatorname{Re}\lambda \leq 0$ that does not contain such exponents. Note that this does not imply $H^{3/2}(\Omega)$ regularity, as it would for a cone with a regular base, because we have strong edge singularities here. The lowest edge exponent is $\lambda^* < 1/3$, see section 3.e *(iii)*, and this corresponds to $H^{1+\sigma}$ regularity for $\boldsymbol{u}$ and H^σ regularity for $\operatorname{div}\boldsymbol{u}$ for all $\sigma < \lambda^*$.

There is, in general, no simple relation between the singular functions of this problem in H_N and those of the Dirichlet or Neumann problems for the Laplace operator. In particular, our explicit constructions in section 7 do not work here, and the classification into types 1, 2, 3 does not make sense. Let us explain two reasons for this.

First, the solutions of the first-order systems (7.2b) and (7.2c) do not belong to H^1 near a non-convex edge. Thus, independently of the corner exponent λ, the X_N-singular functions of all 3 types will not belong to H^1.

Second, the Laplace-Dirichlet problem (7.2a) for q is now posed with only L^2 regularity required. This problem (with $S^\lambda_\tau(\Gamma)$ defined in (6.1))

$$-\Delta q = 0 \ \text{ in } \ \Gamma \quad \text{and} \quad q = 0 \ \text{ on } \partial\Gamma \qquad \text{with } q \in S^{\lambda-1}_0(\Gamma) \tag{8.12}$$

does now not select a discrete set of exponents λ.

Proposition 8.7 *Let Γ be a non-convex pseudo-polyhedral cone. Then the Dirichlet problem* (8.12) *has non-trivial solutions for any $\lambda \in \mathbb{C}$.*

Proof. This is a Laplace-Beltrami eigenvalue problem on $G = \Gamma \cap \mathbb{S}^2$. We look for eigenfunctions in $L^2(G)$. By duality (and the "very weak" definition of the Dirichlet problem (8.12), see (2.9)), we see that such eigenfunctions span the orthogonal complement of the image of $\mathring{H}^1(G) \cap H^2(G)$ under the adjoint operator. Now we know that in the presence of non-convex corners of G, one never has H^2 regularity for this Laplace-Beltrami Dirichlet eigenvalue problem. Thus, in addition to the $\mathring{H}^1(G)$ eigenfunctions that may exist if $\lambda - 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$, we find as many $L^2(G)$ eigenfunctions and therefore solutions to (8.12) as there are non-convex edges meeting at $\boldsymbol{c}$. ■

As a consequence of Proposition 8.7, we lose not only the discreteness of the spectrum $\Lambda^{\mathrm{Dir}}(\Gamma)$, but also the equivalence between our elliptic second order variational formulations with the original Maxwell problem:

- In the presence of non-convex corners, the solutions of problems (1.4) and (1.5), posed in H_N instead of X_N, will depend on the parameter s.
- Even under the hypotheses of Theorem 1.1, the solution of (1.4) will not satisfy $\operatorname{div} \boldsymbol{u} = 0$.
- The solution of (1.5) will not satisfy $p = 0$.

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