A Remark on the Regularity of Solutions of Maxwell's Equations on Lipschitz Domains

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Abstract

Let $\vec{u}$ be a vector field on a bounded Lipschitz domain in $\mathbb{R}^3$, and let $\vec{u}$ together with its divergence and curl be square integrable. If either the normal or the tangential component of $\vec{u}$ is square integrable over the boundary, then $\vec{u}$ belongs to the Sobolev space $H^{1/2}$ on the domain. This result gives a simple explanation for known results on the compact embedding of the space of solutions of Maxwell's equations on Lipschitz domains into L^2.

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with connected Lipschitz boundary Γ. This means that Γ can be represented locally as the graph of a Lipschitz function. For properties of Lipschitz domains, see [7], [3], [2]. In particular, Γ has the strict cone property.

We consider real vector fields $\vec{u}$ on Ω satisfying in the distributional sense

$$\vec{u} \in L^2(\Omega)\,; \quad \operatorname{div} \vec{u} \in L^2(\Omega)\,; \quad \operatorname{curl} \vec{u} \in L^2(\Omega)\,. \tag{1}$$

We denote the inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$.

It is well known that functions $\vec{u}$ satisfying (1) have boundary values $\vec{n} \times \vec{u}$ and $\vec{n} \cdot \vec{u}$ in the Sobolev space $H^{-1/2}(\Gamma)$ defined in the distributional sense by the natural extension of the Green formulas

$$(\operatorname{curl} \vec{u}, \vec{v}) - (\vec{u}, \operatorname{curl} \vec{v}) = <\vec{n} \times \vec{u}, \vec{v}> \tag{2}$$

$$(\operatorname{div} \vec{u}, \varphi) + (\vec{u}, \operatorname{grad} \varphi) = <\vec{n} \cdot \vec{u}, \varphi> \tag{3}$$

for all $\vec{v}, \varphi \in H^1(\Omega)$.

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Here $\vec{n}$ denotes the exterior normal vector which exists almost everywhere on Γ, and $<\cdot, \cdot>$ is the natural duality in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ extending the $L^2(\Gamma)$ inner product.

It is known that for smooth domains (e.g., $\Gamma \in C^{1,1}$), each one of the two boundary conditions

$$\vec{n} \times \vec{u} \in H^{1/2}(\Gamma) \qquad \text{or} \qquad \vec{n} \cdot \vec{u} \in H^{1/2}(\Gamma) \tag{4}$$

implies $\vec{u} \in H^1(\Omega)$, see [2] and, for the case of homogeneous boundary conditions, [6], where one finds also a counterexample for a nonsmooth domain. Such counterexamples are derived from nonsmooth weak solutions $v \in H^1(\Omega)$ of the Neumann problem ($\partial_n := \vec{n} \cdot \operatorname{grad}$ denotes the normal derivative)

$$\Delta v \,=\, g \,\in\, L^2(\Omega)\,; \qquad \partial_n v \,=\, 0 \quad \text{ on } \Gamma \tag{5}$$

If $\vec{u} = \operatorname{grad} v$, then $\vec{u}$ satisfies (1) and $\vec{n} \cdot \vec{u} = 0$ on Γ, and $\vec{u} \in H^s(\Omega)$ if and only if $v \in H^{1+s}(\Omega)$. For smooth or convex domains, one knows that $v \in H^2(\Omega)$. If Ω has a nonconvex edge of opening angle $\alpha\pi$, $\alpha > 1$, then, in general, the solution v of (5) is not in $H^{1+s}(\Omega)$ for $s = 1/\alpha$, hence $\vec{u} \notin H^s(\Omega)$. This upper bound s for the smoothness of $\vec{u}$ can be arbitrary close to $1/2$.

Regularity theorems for (1), (4) have applications in the numerical approximation of the Stokes problem [2] and in the analysis of initial-boundary value problems for Maxwell's equations [6]. The compact embedding into $L^2(\Omega)$ of the space of solutions of the time-harmonic Maxwell equations is needed for the principle of limiting absorption. This compact embedding result was shown by Weck [10] for a class of piecewise smooth domains and by Weber [9] and Picard [8] for general Lipschitz domains. In these proofs, no regularity result for the solution $\vec{u}$ was used or obtained. See Leis' book [6] for a discussion.

In this note, we use the result by Dahlberg, Jerison, and Kenig [4], [5] on the $H^{3/2}$ regularity for solutions of the Dirichlet and Neumann problems with L^2 data in potential theory (see Lemma 1 below). Together with arguments similar to those described by Girault and Raviart [2], this yields $\vec{u} \in H^{1/2}(\Omega)$ (Theorem 2). The compact embedding in L^2 is an obvious consequence of this regularity. If instead of Lemma 1, one uses only the more elementary tools from [1], one obtains $H^{3/2-\epsilon}$ regularity for solutions of the Dirichlet and Neumann problems in potential theory and, consequently $\vec{u} \in H^{1/2-\epsilon}(\Omega)$ for any $\epsilon > 0$. This kind of regularity is also known for the case of an open manifold Γ (screen problem). It suffices, of course, for the compact embedding result.

The proof of the following result can be found in [4].

Lemma 1 *(Dahlberg-Jerison-Kenig) Let* $v \in H^1(\Omega)$ *satisfy* $\Delta v = 0$ *in* Ω. *Then the two conditions*

$$\text{(i)} \quad v\restriction_\Gamma \in H^1(\Gamma) \qquad \textit{and} \qquad \text{(ii)} \quad \partial_n v\restriction_\Gamma \in L^2(\Gamma)$$

are equivalent. They imply $v \in H^{3/2}(\Omega)$.

Remarks.

a.) The first assertion in the Lemma goes back to Nečas [7].

b.) There are accompanying norm estimates, viz.
There exist constants C_1, C_2, C_3, independent of v such that

$$C_1\|\partial_n v\|_{L^2(\Gamma)} \le \|\vec{n} \times \operatorname{grad} v\|_{L^2(\Gamma)} \le C_2\|\partial_n v\|_{L^2(\Gamma)} ,$$

$$\|v\|_{H^{3/2}(\Omega)} \le C_3\|v\restriction_\Gamma \|_{H^1(\Gamma)} .$$

c.) The boundary values are attained in a stronger sense than the distributional sense (2), (3), namely pointwise almost everywhere in the sense of nontangential maximal functions in $L^2(\Gamma)$.

Theorem 2 *Let* $\vec{u}$ *satisfy the conditions* (1) *in* Ω *and either*

$$\vec{n} \times \vec{u} \in L^2(\Gamma) \tag{6}$$

or

$$\vec{n} \cdot \vec{u} \in L^2(\Gamma) . \tag{7}$$

Then $\vec{u} \in H^{1/2}(\Omega)$.
If (1) *is satisfied, then the two conditions* (6) *and* (7) *are equivalent.*

Proof. The proof follows the lines of [2]. It is presented in detail to make sure that it is valid for Lipschitz domains.
Let $\vec{f} := \operatorname{curl}\vec{u} \in L^2(\Omega)$. Then $\operatorname{div}\vec{f} = 0$ in Ω.
According to [2, Ch. I, Thm 3.4] there exists $\vec{w} \in H^1(\Omega)$ with

$$\operatorname{curl}\vec{w} = \vec{f}\,, \quad \operatorname{div}\vec{w} = 0 \quad \text{ in } \Omega. \tag{8}$$

The construction of $\vec{w}$ is as follows:
Choose a ball $\mathcal{O}$ containing $\overline{\Omega}$ in its interior and solve in $\mathcal{O} \setminus \overline{\Omega}$ the Neumann problem: $\chi \in H^1(\mathcal{O} \setminus \overline{\Omega})$ with

$$\Delta\chi = 0 \text{ in } \mathcal{O} \setminus \overline{\Omega}\,;\ \partial_n\chi = \vec{n} \cdot \vec{f} \text{ on } \Gamma\,;\ \partial_n\chi = 0 \text{ on } \partial\mathcal{O}\,. \tag{9}$$

Note that $\vec{n} \cdot \vec{f} \in H^{-1/2}(\Gamma)$ satisfies the solvability condition $<\vec{n} \cdot \vec{f}, 1> = 0$ because $\operatorname{div}\vec{f} = 0$ in Ω.

Define $\vec{f}_0 := \vec{f}$ in Ω, $\vec{f}_0 := \operatorname{grad}\chi$ in $\mathcal{O}\backslash\overline{\Omega}$, $\vec{f}_0 := 0$ in $\mathbb{R}^3\backslash\overline{\mathcal{O}}$. Then $\vec{f}_0 \in L^2(\mathbb{R}^3)$ has compact support and satisfies $\operatorname{div}\vec{f}_0 = 0$ in $\mathbb{R}^3$. Therefore $\vec{f}_0 = \operatorname{curl}\vec{w}$ for some $\vec{w} \in H^1(\mathbb{R}^3)$ with $\operatorname{div}\vec{w} = 0$ in $\mathbb{R}^3$. One obtains $\vec{w}$ for example by convolution of $\vec{f}_0$ with a fundamental solution of the Laplace operator in $\mathbb{R}^3$ and taking the curl.

Thus (8) is satisfied. The function $\vec{z} := \vec{u} - \vec{w}$ satisfies

$$\vec{z} \in L^2(\Omega) \quad \text{and} \quad \operatorname{curl}\vec{z} = 0 \quad \text{in } \Omega. \tag{10}$$

Since Ω is simply connected, there exists $v \in H^1(\Omega)$ with

$$\vec{z} \;=\; \operatorname{grad} v \;. \tag{11}$$

Then v satisfies

$$\Delta v \;=\; \operatorname{div}\vec{u} \;\in\; L^2(\Omega)\;. \tag{12}$$

We can apply Lemma 1 to v, because by subtraction of a suitable function in $H^2(\Omega)$, we obtain a homogeneous Laplace equation from (12).

Now, since $\vec{w}\restriction_\Gamma \in H^{1/2}(\Gamma)$, condition (i) in the Lemma is equivalent to

$$\vec{n} \times \operatorname{grad} v = \vec{n} \times \vec{z} = \vec{n} \times \vec{u} - \vec{n} \times \vec{w} \in L^2(\Gamma)$$

and hence to (6), and condition (ii) is equivalent to

$$\vec{n} \cdot \operatorname{grad} v = \vec{n} \cdot \vec{z} = \vec{n} \cdot \vec{u} - \vec{n} \cdot \vec{w} \in L^2(\Gamma)$$

and hence to (7). Therefore the Lemma implies that (6) and (7) are equivalent.
Also, $v \in H^{3/2}(\Omega)$ is equivalent to $\operatorname{grad} v \in H^{1/2}(\Omega)$, hence to

$$\vec{u} = \vec{z} + \vec{w} = \operatorname{grad} v + \vec{w} \in H^{1/2}(\Omega)\;.$$

∎

Remark. The accompanying norm estimates are:
There exist constants C_1, C_2, C_3, independent of $\vec{u}$ such that

$$\begin{aligned}
\|\vec{n} \times \vec{u}\|_{L^2(\Gamma)} &\leq C_1 \left(\|\vec{u}\|_{L^2(\Omega)} + \|\operatorname{div}\vec{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\vec{u}\|_{L^2(\Omega)} + \|\vec{n} \cdot \vec{u}\|_{L^2(\Gamma)}\right) \\
\|\vec{n} \cdot \vec{u}\|_{L^2(\Gamma)} &\leq C_2 \left(\|\vec{u}\|_{L^2(\Omega)} + \|\operatorname{div}\vec{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\vec{u}\|_{L^2(\Omega)} + \|\vec{n} \times \vec{u}\|_{L^2(\Gamma)}\right) \\
\|\vec{u}\|_{H^{1/2}(\Omega)} &\leq C_3 \left(\|\vec{u}\|_{L^2(\Omega)} + \|\operatorname{div}\vec{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\vec{u}\|_{L^2(\Omega)} + \|\vec{n} \times \vec{u}\|_{L^2(\Gamma)}\right)\;.
\end{aligned}$$

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